# AN EXAMPLE OF THE LANGLANDS CORRESPONDENCE FOR IRREGULAR RANK TWO CONNECTIONS ON $\mathbb{P}^1$.

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Abstract. Special kinds of rank 2 vector bundles with (possibly irregular) connections on $\mathbb{P}^1$ are considered. We construct an equivalence between the derived category of quasi-coherent sheaves on the moduli stack of such bundles and the derived category of modules over a TDO ring on certain non-separated curve. We identify this curve with the coarse moduli space of some parabolic bundles on $\mathbb{P}^1$. Then our equivalence becomes an example of the categorical Langlands correspondence.

## 1. Introduction

Let $\mathcal{C}onn(X, r)$ be the moduli space of rank $r$ vector bundles with connections on a smooth complex projective curve $X$. Let $\mathcal{B}un(X, r)$ be the moduli space of rank $r$ vector bundles on $X$. The *categorical Langlands correspondence* for $GL(r)$ is a conjectural equivalence between the derived category of $O$-modules on $\mathcal{C}onn(X, r)$ and the derived category of $\mathcal{D}$-modules on $\mathcal{B}un(X, r)$. We refer the interested reader to [Fre1, §6.2].

This correspondence has been proved by one of the authors in the settings of rank two bundles equipped with connections with four simple poles on $X = \mathbb{P}^1$ (cf. [Ari2]). In this case, the space $\mathcal{B}un(X, r)$ should be replaced by the moduli space of bundles with parabolic structures. More precisely, [Ari2] works with $SL(2)$-connections and $PGL(2)$-bundles. (See [Fre2] for a discussion of the ramified Langlands program.)

In this paper we extend the results of [Ari2] to the case when the ramification divisor still has degree four but we allow higher order poles as long as leading terms are regular semisimple (see Theorems 3 and 2). This provides an example of the categorical Langlands correspondence for connections with irregular singularities.

In [FG], Frenkel and Gross present an example of the Langlands correspondence for a different kind of irregular singularities. It is instructive to compare the two settings. Unlike the present paper, the results of Frenkel and Gross apply to arbitrary group $G$, not just $G = GL(2)$. The ramification considered is in a sense the simplest nontrivial: the ramification divisor has degree three. It is proved in [FG] that in these settings, there is a unique up to isomorphism local system with prescribed singularities. In other words, the counterpart of the moduli space $\mathcal{C}onn(X, r)$ consists of a single point. In particular, the category of $O$-modules on this space has a unique irreducible object, the structure sheaf of this point. The corresponding category of automorphic $\mathcal{D}$-modules (the counterpart of the category of $\mathcal{D}$-modules on $\mathcal{B}un(X, r)$) also has a unique irreducible object ([FG] Sections 3, 16).

*Key words and phrases.* the Langlands duality; Moduli spaces; Connections with irregular singularities; Parabolic bundles.

The categorical Langlands transform sends the two irreducible generators into each other.

The present paper studies the 'next simplest case': the ramification divisor has degree four. The moduli space of local systems is a surface, and the categorical Langlands transform is an equivalence, similar to the Fourier-Mukai transform.

The techniques used in our argument are similar to that of [Ari2] but more conceptual. We hope that the present proof is more suitable for generalizations to divisors of higher order and to the higher genus case.

*Remark.* In positive characteristic, a different approach to Langlands correspondence was discovered by Bezrukavnikov and Braverman. In [BB2], they construct a version of the categorical Langlands correspondence. In [Nev], Nevins uses these ideas for connections with regular singularities.

Our argument requires two steps that may be of independent interest. Firstly, in §3 we prove that the moduli space of connections with possibly irregular singularities has a good moduli space in the sense of [Alp]; we also construct a modular projective compactification of this space, see Theorems 6 and 7. This is an extension of Simpson's results [Sim1, Sim2, Sim3].

Secondly, in §5 we study the compactified Jacobians of singular degenerations of elliptic curves, see Proposition 5.7, and construct a Fourier-Mukai transform, see Theorem 8.

Finally, we want to note that our moduli spaces of connections are the moduli spaces of initial conditions of Painlevé equations. More precisely, the case of regular singularities corresponds to Painlevé VI, while the cases of irregular singularities correspond to Painlevé II–Painlevé V, see [OO].

1.1. **Conventions.** We work over the ground field of complex numbers, thus $\mathbb{P}^1$ means $\mathbb{P}^1_{\mathbb{C}}$, a 'scheme' means a '$\mathbb{C}$-scheme' etc. All schemes and stacks are locally of finite type.

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## 2. Main Results

Let $\mathfrak{D} := \sum n_i x_i$ be a divisor on $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}}$ with $n_i > 0$. Let $L$ be a rank 2 vector bundle on $\mathbb{P}^1$, $\nabla : L \to L \otimes \Omega_{\mathbb{P}^1}(\mathfrak{D})$ a connection on $L$ with polar divisor $\mathfrak{D}$. We call such pairs $(L, \nabla)$ *connections* for brevity.

Choosing a formal coordinate $z$ near $x_i$ and a trivialization of $L$ on the formal neighborhood of $x_i$, we can write $\nabla$ near $x_i$ as

$$\mathbf{d} + a\frac{\mathbf{d}z}{z^{n_i}} + \text{higher order terms}, \qquad a \in \mathfrak{gl}(2).$$

In the case $n_i > 1$ the connection will be called *non-resonant* at $x_i$ if $a$ has distinct eigenvalues; in the case $n_i = 1$ the connection will be called non-resonant if the eigenvalues do not differ by an integer. The connection will be called non-resonant if it is non-resonant at all $x_i$. Up to scaling, the conjugacy class of $a$ does not depend

on the choices. Thus the notion of non-resonant connections does not depend on the choices.

2.1. **Moduli stacks.** Let $(L, \nabla)$ be a non-resonant connection, then in a suitable trivialization of $L$ over the formal disc at $x_i$ the connection takes a diagonal form

$$\nabla = \mathbf{d} + \begin{pmatrix} \alpha_i^+ & 0 \\ 0 & \alpha_i^- \end{pmatrix}, \tag{2.1}$$

where $\alpha_i^\pm$ are 1-forms on the formal disc. The polar parts of these 1-forms do not depend on the trivialization of $L$, thus we shall call them *the formal type of* $\nabla$ *at* $x_i$.

Fix $\mathfrak{D}$ and for each $i$ polar parts $\alpha_i^\pm$ of 1-forms at $x_i$. Assume that these polar parts satisfy the following conditions

(a) The order of the pole of $\alpha_i^\pm$ is at most $n_i$, the order of the pole of $\alpha_i^+ - \alpha_i^-$ is exactly $n_i$.

(b) $d := -\sum_i \operatorname{res}(\alpha_i^+ + \alpha_i^-)$ is integer.

(c) $\sum_i \operatorname{res} \alpha_i^\pm \notin \mathbb{Z}$. (Here for each $i$ there is exactly one summand $\alpha_i^\pm$, and the choices of signs $+$ and $-$ are independent.)

(d) If $n_i = 1$, then $\operatorname{res} \alpha_i^+ - \operatorname{res} \alpha_i^- \notin \mathbb{Z}$.

Let $\mathcal{M} = \mathcal{M}(\mathfrak{D}, \alpha_i^\pm)$ be the moduli space of connections $(L, \nabla)$ such that $\nabla$ has formal types $\alpha_i^\pm$ at $x_i$. Note that such a connection is non-resonant by (a) and (d). Also, $L$ has degree $d$ by (b).

From now on we assume that $\deg \mathfrak{D} = 4$.

**Theorem 1.** *The moduli space $\mathcal{M}$ is a smooth connected algebraic stack of dimension* 1*. It is a neutral $\mathbf{G_m}$-gerbe over its coarse moduli space $M$; besides, $M$ is a smooth quasi-projective surface.*

This theorem will be proved in §4.

Let $\operatorname{Qcoh}(\mathcal{M})$ be the category of quasi-coherent sheaves on $\mathcal{M}$. Since $\mathcal{M}$ is a $\mathbf{G_m}$-gerbe, we obtain a decomposition

$$\operatorname{Qcoh}(\mathcal{M}) = \prod_{i \in \mathbb{Z}} \operatorname{Qcoh}(\mathcal{M})^{(i)},$$

where $\mathcal{F} \in \operatorname{Qcoh}(\mathcal{M})^{(i)}$ if $t \in \mathbf{G_m}$ acts on $\mathcal{F}$ as $t^i$. Let $\mathcal{D}^b(\mathcal{M})$ be the corresponding bounded derived category. By definition, objects of $\mathcal{D}^b(\mathcal{M})$ are complexes of $O_{\mathcal{M}}$-modules with quasi-coherent cohomology. It follows from [AB, Claim 2.7] that $\mathcal{D}^b(\mathcal{M})$ is equivalent to the bounded derived category of $\operatorname{Qcoh}(\mathcal{M})$. Thus we also have a decomposition

$$\mathcal{D}^b(\mathcal{M}) = \prod_{i \in \mathbb{Z}} \mathcal{D}^b(\mathcal{M})^{(i)}.$$

It is easy to see that $\mathcal{F} \in \mathcal{D}^b(\mathcal{M})^{(i)}$ if and only if $H^\bullet(\mathcal{F}) \in \operatorname{Qcoh}(\mathcal{M})^{(i)}$.

2.2. **Twisted differential operators.** Denote by $\wp : P \to \mathbb{P}^1$ the projective line with points $x_i$ doubled. In other words, $P$ is obtained by gluing two copies of $\mathbb{P}^1$ outside the support of $\mathfrak{D}$. Denote the preimages of $x_i$ by $x_i^-$ and $x_i^+$. Let $j : \mathbb{P}^1 - \mathfrak{D} \hookrightarrow P$ be the natural embedding. This notation will be used throughout the paper.

The main result of the present paper is that $\mathcal{D}^b(\mathcal{M})^{(-1)}$ is equivalent to a category of twisted $\mathcal{D}$-modules on $P$. To give a precise definition of this twist, recall that the isomorphism classes of sheaves of rings of twisted differential operators

(TDO) on a smooth (not necessarily separated) curve are classified by the first cohomology group of the sheaf of 1-forms.

**Lemma 2.1.** *Denote by $\omega_i$ the vector space of polar parts of 1-forms at $x_i \in \mathbb{P}^1$. Then*

$$H^1(P, \Omega_P) = \mathbb{C} \oplus \bigoplus_i \omega_i.$$

*Proof.* Let $D_i^{\pm}$ be the formal disc at $x_i^{\pm}$. These discs, together with $\mathbb{P}^1 - \mathfrak{D}$, give a cover of $P$; let us use the corresponding Čech complex. We see that a 1-cocycle is a collection $\beta_i^{\pm}$ of 1-forms on punctured formal discs, and one easily checks that the map $(\beta_i^{\pm}) \mapsto \left(\sum_i \operatorname{res}(\beta_i^+ + \beta_i^-), \beta_i^+ - \beta_i^-\right)$ induces the required isomorphism. □

Using this lemma, we define the sheaf of differential operators on $P$ twisted by $(-d, \alpha_i^+ - \alpha_i^-)$; denote it by $\mathcal{D}_{P,\alpha}$. In other words, it is given by the 1-cocycle $(\alpha_i^{\pm})$.

2.3. **The integral transform.** Let $\xi = (L, \nabla) \in \mathcal{M}$. Denote by $\xi_\alpha$ the $\mathcal{D}_{P,\alpha}$-module generated by $\wp^*\xi$. More precisely, $\xi_\alpha := j_{!*}(\xi|_{\mathbb{P}^1 - \mathfrak{D}})$, where $j_{!*}$ is the middle extension for $\mathcal{D}_{P,\alpha}$-modules. Since $\xi|_{\mathbb{P}^1 - \mathfrak{D}}$ is a $\mathcal{D}_{\mathbb{P}^1 - \mathfrak{D}}$-module, and the twist of $\mathcal{D}_{P,\alpha}$ is supported outside of $\mathbb{P}^1 - \mathfrak{D}$, $\xi_\alpha$ is well defined.

*Remark* 2.2. Let us describe the restriction of $\xi_\alpha$ to $\wp^{-1}(D_i)$, where $D_i$ is the formal disc centered at $x_i$. Choose 1-forms $\tilde{\alpha}^{\pm}$ with polar parts $\alpha_i^{\pm}$. According to (2.1), the restriction of $\xi$ to $D_i$ is isomorphic to

$$(O_{D_i}, \mathbf{d} + \tilde{\alpha}^-) \oplus (O_{D_i}, \mathbf{d} + \tilde{\alpha}^+).$$

Now one checks easily that

$$\xi_\alpha|_{D_i^-} \simeq (O_{D_i^-}, \mathbf{d} + \tilde{\alpha}^-) \oplus (O_{\dot{D}}, \mathbf{d} + \tilde{\alpha}^+),$$
$$\xi_\alpha|_{D_i^+} \simeq (O_{\dot{D}}, \mathbf{d} + \tilde{\alpha}^-) \oplus (O_{D_i^+}, \mathbf{d} + \tilde{\alpha}^+),$$

where $\dot{D} \subset D_i^{\pm}$ is the punctured formal disc.

Note that formal normal form exists for families of connections, and it is constant for families in $\mathcal{M}$. Thus our middle extension construction still makes sense for families of connections in $\mathcal{M}$. Hence we can apply it to the universal family $\xi$ on $\mathcal{M} \times \mathbb{P}^1$, getting an $\mathcal{M}$-family $\xi_\alpha$ of $\mathcal{D}_{P,\alpha}$-modules. In other words, $\xi_\alpha$ is an $O_{\mathcal{M}} \boxtimes \mathcal{D}_{P,\alpha}$-module on $\mathcal{M} \times P$. Thus $\xi_\alpha$ gives rise to an integral transform from $\mathcal{D}^b(\mathcal{M})$ to the derived category of $\mathcal{D}_{P,\alpha}$-modules.

Denote the natural projections of $\mathcal{M} \times P$ to $\mathcal{M}$ and $P$ by $p_1$ and $p_2$ respectively.

**Theorem 2.** *Let $d$ be an odd number. Then the functor*

$$\Phi_{\mathcal{M}\to P} : \mathcal{F} \mapsto Rp_{2,*}\Big(\xi_\alpha \underset{O_{\mathcal{M}\times P}}{\otimes} p_1^*\mathcal{F}\Big)$$

*is an equivalence between $\mathcal{D}^b(\mathcal{M})^{(-1)}$ and the bounded derived category of $\mathcal{D}_{P,\alpha}$-modules.*

Theorem 2 is the main result of the paper; we prove it in §§2.5–8.

*Remark* 2.3. (a) It is easy to see that the restriction of $\Phi_{\mathcal{M}\to P}$ to $\mathcal{D}^b(\mathcal{M})^{(i)}$ is zero unless $i = -1$.
(b) On the other hand, $\mathcal{D}^b(\mathcal{M})^{(i)}$ depends only on parity of $i$. Indeed, fix $x \in \mathbb{P}^1 - \mathfrak{D}$, and let $\delta_x$ be the line bundle on $\mathcal{M}$ whose fiber at $(L, \nabla)$ is equal to $\det L_x$. Then the tensor product with $\delta_x$ provides an equivalence between $\mathcal{D}^b(\mathcal{M})^{(i)}$ and $\mathcal{D}^b(\mathcal{M})^{(i+2)}$.

(c) Assume that $\mathfrak{D} = \sum n_i x_i$ is not even, that is one of the numbers $n_i$ is odd, then all the categories $\mathcal{D}^b(\mathcal{M})^{(i)}$ are equivalent. Indeed, let $n_i$ be odd, and for $(L, \nabla) \in \mathcal{M}$ let $\eta_i$ be a unique level $n_i$ parabolic structure at $x_i$ compatible with $\nabla$ (see Definition 2.4 and §4.3). Tensoring with the line bundle whose fiber at $(L, \nabla)$ is $\det \eta_i$, we get an equivalence between the odd and the even components of the derived category.
(d) In fact our theorem is also valid if $d$ is even but $\mathfrak{D}$ is not even. Indeed, let $n_i$ be odd and define a collection $\beta_i^\pm$ of polar parts of 1-forms by

$$\beta_i^+ = \alpha_i^+ + n_i \frac{\mathbf{d}z}{z}, \qquad \beta_i^- = \alpha_i^-, \qquad \beta_j^\pm = \alpha_j^\pm \text{ for } i \neq j.$$

Then a modification at $x_i$ provides an isomorphism $\mathcal{M}(\alpha) \simeq \mathcal{M}(\beta)$, and we can apply the theorem to $\mathcal{M}(\beta)$. (See §4.4 for the definition of modification.) It remains to notice that the category of $\mathcal{D}_{P,\alpha}$-modules is equivalent to the category of $\mathcal{D}_{P,\beta}$-modules: the equivalence is given by tensoring with a line bundle.

## 2.4. The Langlands Correspondence.

**Definition 2.4.** Let $L$ be a rank 2 vector bundle on $\mathbb{P}^1$. A *level-$\mathfrak{D}$ parabolic structure* on $L$ is a line subbundle $\eta$ in the restriction of $L$ to $\mathfrak{D}$ (we view $\mathfrak{D}$ as a non-reduced subscheme of $\mathbb{P}^1$). We call a bundle with a parabolic structure *a parabolic bundle.*

Let $\overline{\mathcal{B}un}(d^\vee) = \overline{\mathcal{B}un}(\mathbb{P}^1, 2, d^\vee, \mathfrak{D})$ be the moduli stack of rank 2 degree $d^\vee$ vector bundles on $\mathbb{P}^1$ with level-$\mathfrak{D}$ parabolic structures. (We reserve notation $\mathcal{B}un$ for its open substack of bundles without non-scalar endomorphisms, cf. §4.3.)

Let $\mathbb{C}[\mathfrak{D}]$ be the ring of functions on the scheme $\mathfrak{D}$, $\mathbb{C}[\mathfrak{D}]^\times$ be the group of invertible functions; this is an algebraic group. Choosing local coordinates at the points $x_i$, we get an isomorphism

$$\mathbb{C}[\mathfrak{D}]^\times = \prod_i (\mathbb{C}[z]/z^{n_i})^\times.$$

Let $\pi : \eta_{univ} \to \overline{\mathcal{B}un}(d^\vee)$ be the $\mathbb{C}[\mathfrak{D}]^\times$-torsor whose fiber over $(L, \eta) \in \overline{\mathcal{B}un}(d^\vee)$ is

$$\{s \in H^0(\mathfrak{D}, \eta) |\ s(x_i) \neq 0 \text{ for all } i\}.$$

The collection $\alpha_i^+$ of polar parts of 1-forms can be viewed as an element of

$$(\mathrm{Lie}(\mathbb{C}[\mathfrak{D}]^\times))^\vee = \mathbb{C}[\mathfrak{D}]^\vee$$

via the residue pairing. Thus it gives rise to a TDO ring on $\overline{\mathcal{B}un}(d^\vee)$ through non-commutative reduction of the sheaf of differential operators on the total space of $\eta_{univ}$ (see §9.1). We denote this TDO ring by $\mathcal{D}_{\overline{\mathcal{B}un}(d^\vee),\alpha^+}$.

Similarly, we define a $\mathbb{C}[\mathfrak{D}]^\times$-torsor $\eta'_{univ}$ whose fiber over $(L, \eta) \in \overline{\mathcal{B}un}(d^\vee)$ is

$$\{s \in H^0(\mathfrak{D}, (L|_\mathfrak{D})/\eta) |\ s(x_i) \neq 0 \ \text{ for all } i\}.$$

Denote the TDO ring corresponding to $\eta'_{univ}$ and the collection $\alpha_i^-$ by $\mathcal{D}_{\overline{\mathcal{B}un}(d^\vee),\alpha^-}$. Let $\mathcal{D}_{\overline{\mathcal{B}un}(d^\vee),\alpha}$ be the Baer sum of the TDO rings $\mathcal{D}_{\overline{\mathcal{B}un}(d^\vee),\alpha^+}$ and $\mathcal{D}_{\overline{\mathcal{B}un}(d^\vee),\alpha^-}$.

**Theorem 3** (the Langlands correspondence)**.** *Assume that $d$ is an odd number. Then $\mathcal{D}^b(\mathcal{M})^{(-1)}$ is equivalent to the bounded derived category of $\mathcal{D}_{\overline{\mathcal{B}un}(-1),\alpha}$-modules.*

Theorem 3 is derived from Theorem 2 in §9.

*Remark* 2.5. (a) Let us discuss the notion of the derived category of $\mathcal{D}_{\overline{\mathcal{B}un}(-1),\alpha}$-modules. Note that $\overline{\mathcal{B}un}(-1)$ is a (smooth) algebraic stack, so this notion is not immediate.

As we show in §9 the category of $\mathcal{D}_{\overline{\mathcal{B}un}(-1),\alpha}$-modules is equivalent to a category of twisted $\mathcal{D}$-modules on $P$ (in fact $P$ is the coarse moduli space of a certain open subset of $\overline{\mathcal{B}un}(-1)$). Thus we shall view the derived category of twisted $\mathcal{D}$-modules on $P$ as the definition for the derived category of $\mathcal{D}_{\overline{\mathcal{B}un}(-1),\alpha}$-modules. (See also the discussion in §9.)

(b) In general, we expect an equivalence of categories between $\mathcal{D}^b(\mathcal{M})^{(d^\vee)}$ and the bounded derived category of $\mathcal{D}_{\overline{\mathcal{B}un}(d^\vee),\alpha}$-modules. This statement follows from Theorem 3 if $d^\vee$ is odd. Indeed, pick $x \in \mathbb{P}^1 - \mathfrak{D}$, then $(L,\eta) \mapsto (L(x),\eta)$ is an isomorphism between $\overline{\mathcal{B}un}(d^\vee)$ and $\overline{\mathcal{B}un}(d^\vee+2)$. It remains to use Remark 2.3(b).

(c) We also have the desired equivalence if $\mathfrak{D}$ is not even. Indeed, let $n_i$ be odd. Modification at $x_i$ gives an isomorphism $\mathcal{D}_{\overline{\mathcal{B}un}(d^\vee),\alpha} \simeq \mathcal{D}_{\overline{\mathcal{B}un}(d^\vee+n_i),\beta}$, where $\beta$ is obtained from $\alpha$ by swapping $\alpha_i^+$ and $\alpha_i^-$. It remains to use the previous remark, Remark 2.3(c), and the obvious identification $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$.

2.5. **The plan of proof of Theorem 2.** Theorem 2 reduces to two orthogonality statements.

Let $p_{12} : P \times P \times \mathcal{M} \to P \times P$ and $p_{13}, p_{23} : P \times P \times \mathcal{M} \to P \times \mathcal{M}$ be the projections. Let $\xi^\vee$ be the vector bundle on $\mathbb{P}^1 \times \mathcal{M}$ dual to $\xi$. Since it has a connection along $\mathbb{P}^1$, we see that $\xi_\alpha^\vee := (\xi^\vee)_{-\alpha}$ is a $\mathcal{D}_{P,-\alpha} \boxtimes O_{\mathcal{M}}$-module on $P \times \mathcal{M}$.

Set $\mathcal{F}_P := (p_{13}^*\xi_\alpha) \otimes (p_{23}^*\xi_\alpha^\vee)$. Here $p_{13}^*$ and $p_{23}^*$ stand for the $O$-module pullback (from the viewpoint of $\mathcal{D}$-modules, these pullback functors should include a cohomological shift). Note that $\xi_\alpha$ is a flat $O_{P\times\mathcal{M}}$-module (see Remark 2.2), hence

$$(p_{13}^*\xi_\alpha) \otimes (p_{23}^*\xi_\alpha^\vee) = (p_{13}^*\xi_\alpha) \otimes^L (p_{23}^*\xi_\alpha^\vee).$$

Further, $Rp_{12,*}\mathcal{F}_P$ is an object of the derived category of $p_1^\bullet\mathcal{D}_{P,\alpha} \circledast p_2^\bullet\mathcal{D}_{P,-\alpha}$-modules, where $p_1, p_2 : P \times P \to P$ are the projections. Here $p_i^\bullet$ (resp. $\circledast$) stands for the inverse image (resp. Baer sum) of TDO rings (the corresponding functors on Lie algebroids are described in [BB1]).

**Theorem 4.** *$Rp_{12,*}\mathcal{F}_P = \delta_\Delta[-1]$, where $\Delta \subset P \times P$ is the diagonal, and $\delta_\Delta$ is the direct image of $O_\Delta$ as a $\mathcal{D}_\Delta$-module.*

*Remark* 2.6. In general, for a map $f : X \to Y$ and a TDO ring $\mathcal{D}_1$ on $Y$, there is a functor $f_+ : \mathcal{D}^b(f^\bullet\mathcal{D}_1) \to \mathcal{D}^b(\mathcal{D}_1)$, where $\mathcal{D}^b(\mathcal{D}_1)$ is the bounded derived category of $\mathcal{D}_1$-modules. For the embedding $i : \Delta \hookrightarrow P \times P$, one easily checks that $i^\bullet(p_1^\bullet\mathcal{D}_{P,\alpha} \circledast p_2^\bullet\mathcal{D}_{P,-\alpha})$ is the (non-twisted) differential operator ring $\mathcal{D}_\Delta$, so $\delta_\Delta := i_+(O_\Delta)$ is well defined as a $p_1^\bullet\mathcal{D}_{P,\alpha} \circledast p_2^\bullet\mathcal{D}_{P,-\alpha}$-module.

By Theorem 4, $\xi_\alpha$ is an orthogonal $P$-family of vector bundles on $\mathcal{M}$. To obtain an equivalence of categories, one should also show that $\xi_\alpha$ is orthogonal as an $\mathcal{M}$-family of $\mathcal{D}_{P,\alpha}$-modules. Let us give the precise statement. We follow closely S. Lysenko's unpublished notes [Lys].

Consider $\mathcal{F}_{\mathcal{M}} := p_{13}^*\xi_\alpha \otimes p_{23}^*\xi_\alpha^\vee$ (here $p_{13}, p_{23} : \mathcal{M} \times \mathcal{M} \times P \to \mathcal{M} \times P$ are the projections).

$\mathcal{F}_{\mathcal{M}}$ can be viewed as a family of $\mathcal{D}_P$-modules parameterized by $\mathcal{M} \times \mathcal{M}$. Consider the de Rham complex of $\mathcal{F}_{\mathcal{M}}$ in the direction of $P$

$$\mathbb{DR}(\mathcal{F}_{\mathcal{M}}) = \mathbb{DR}_P(\mathcal{F}_{\mathcal{M}}) := (\mathcal{F}_{\mathcal{M}} \to \mathcal{F}_{\mathcal{M}} \otimes \Omega_{\mathcal{M}\times\mathcal{M}\times P/\mathcal{M}\times\mathcal{M}}).$$

Our aim is to compute $Rp_{12,*}\,\mathbb{DR}(\mathcal{F}_{\mathcal{M}})$.

By Theorem 1, $\mathcal{M}\times\mathcal{M}$ is a $\mathbf{G_m}\times\mathbf{G_m}$-gerbe over a scheme, so $\mathbf{G_m}\times\mathbf{G_m}$ acts on any quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{M}$. Therefore, $\mathcal{F}$ can be decomposed with respect to the characters of $\mathbf{G_m}\times\mathbf{G_m}$. Denote by $\mathcal{F}^\psi$ the component of $\mathcal{F}$ corresponding to the character $\psi:\mathbf{G_m}\times\mathbf{G_m}\to\mathbf{G_m}$ defined by $(t_1,t_2)\mapsto t_1/t_2$.

Let $\operatorname{diag}:\mathcal{M}\to\mathcal{M}\times\mathcal{M}$ be the diagonal morphism.

**Theorem 5** (S. Lysenko)**.** $Rp_{12,*}\,\mathbb{DR}(\mathcal{F}_{\mathcal{M}})=(\operatorname{diag}_*O_{\mathcal{M}})^\psi[-2]$.

*Remark* 2.7. Note that $\mathcal{M}$ is a $\mathbf{G_m}$-torsor over $\operatorname{diag}(\mathcal{M})$. Thus

$$\operatorname{diag}_*O_{\mathcal{M}}=\bigoplus_{i\in\mathbb{Z}}(\operatorname{diag}_*O_{\mathcal{M}})^{\psi^i}.$$

Objects of $\mathcal{D}^b(\mathcal{M}\times\mathcal{M})$ define endofunctors on $\mathcal{D}^b(\mathcal{M})$. Let us consider the functors corresponding to the components of $\operatorname{diag}_*O_{\mathcal{M}}$. Clearly,

$$\mathcal{D}^b(\mathcal{M})\to\mathcal{D}^b(\mathcal{M}):\mathcal{F}\mapsto Rp_{2,*}((\operatorname{diag}_*O_{\mathcal{M}})\otimes p_1^*\mathcal{F})$$

is isomorphic to the identity functor. It is easy to see that the functor

$$\mathcal{D}^b(\mathcal{M})\to\mathcal{D}^b(\mathcal{M}):\mathcal{F}\mapsto Rp_{2,*}((\operatorname{diag}_*O_{\mathcal{M}})^{\psi^i}\otimes p_1^*\mathcal{F})$$

is isomorphic to the projection $\mathcal{D}^b(\mathcal{M})\to\mathcal{D}^b(\mathcal{M})^{(-i)}$.

*Proof of Theorem 2.* The inverse to the functor $\Phi_{\mathcal{M}\to P}$ is given by

$$\Phi_{P\to\mathcal{M}}:\mathcal{F}\mapsto Rp_{1,*}\,\mathbb{DR}(\xi_\alpha^\vee\otimes p_2^*\mathcal{F})[2].$$

Indeed, using base change and Theorem 4, one checks that the composition $\Phi_{\mathcal{M}\to P}\circ\Phi_{P\to\mathcal{M}}$ is isomorphic to the identity functor. Similarly, it follows from Theorem 5 and Remark 2.7 that the composition $\Phi_{P\to\mathcal{M}}\circ\Phi_{\mathcal{M}\to P}$ is isomorphic to the projection $\mathcal{D}^b(\mathcal{M})\to\mathcal{D}^b(\mathcal{M})^{(-1)}$. □

## 3. A compactification of moduli spaces of connections

In this section, we compactify a moduli space of connections with singularities following C. Simpson ([Sim1, Sim2, Sim3]).

In [Sim3], C. Simpson constructs a natural compactification of the moduli space of vector bundles with connections on a smooth projective variety $X$. We consider the case when $X$ is a smooth projective curve. In this case it is not hard to generalize the result to the case of connections with singularities (we use [Sim2]). Then we prove that the compactification is in fact projective (note that for varieties of higher dimension projectivity of the compactification is not known). Our description of the divisor at infinity is also more explicit than in [Sim3].

The compactification is constructed in the following generality. Let $X$ be a smooth complex projective curve, $r$ a positive integer, $d$ an integer, and $\mathfrak{D}$ an effective divisor on $X$. Denote by $\mathcal{N}=\mathcal{N}(X,r,d,\mathfrak{D})$ the moduli stack of pairs $(L,\nabla)$, where $L$ is a vector bundle on $X$ of rank $r$ and degree $d$, and $\nabla:L\to L\otimes\Omega_X(\mathfrak{D})$ is a connection on $L$ with the order of poles bounded by $\mathfrak{D}$. Our goal is to construct a compactification of the semistable part of $\mathcal{N}$.

Fix $X$, $r$, $d$, and $\mathfrak{D}$.

3.1. **$\varepsilon$-connections.** The compactification is constructed as a moduli space of P. Deligne's $\lambda$-connections. Recall the following

**Definition 3.1.** Let $L$ be a vector bundle on $X$. For a one-dimensional vector space $E$ and $\varepsilon \in E$, an *$\varepsilon$-connection* on $L$ is a $\mathbb{C}$-linear map $\nabla : L \to L \otimes \Omega_X \otimes_{\mathbb{C}} E$ such that

$$\nabla(fs) = f\nabla s + s \otimes \mathbf{d}f \otimes \varepsilon \text{ for all } f \in O_X, s \in L.$$

More generally, an $\varepsilon$-connection on $L$ with poles bounded by $\mathfrak{D}$ is a map $\nabla : L \to L \otimes \Omega_X(\mathfrak{D}) \otimes_{\mathbb{C}} E$ satisfying the same condition.

Denote by $\overline{\mathcal{N}} = \overline{\mathcal{N}}(X, r, d, \mathfrak{D})$ the moduli stack of collections $(L, \nabla; \varepsilon \in E)$, where $L$ is a vector bundle on $X$ of rank $r$ and degree $d$, and $\nabla$ is an $\varepsilon$-connection on $L$ with poles bounded by $\mathfrak{D}$. This is an algebraic stack, the proof is similar to [Fed, Proposition 1].

**Definition 3.2.** $(L, \nabla; \varepsilon \in E) \in \overline{\mathcal{N}}$ is *semistable* if for any non-zero $\nabla$-invariant subbundle $L_0 \subset L$ we have

$$\frac{\deg L_0}{\operatorname{rk} L_0} \leq \frac{d}{r}.$$

Further, $(L, \nabla; \varepsilon \in E)$ is *nilpotent* if $\varepsilon = 0$ and $\nabla^r = 0$. Note that if $\varepsilon = 0$, $\nabla$ is $O_X$-linear, so $\nabla^r$ makes sense as a map $L \to L \otimes (\Omega_X(\mathfrak{D}) \otimes_{\mathbb{C}} E)^{\otimes r}$. Equivalently, $(L, \nabla; \varepsilon \in E)$ is nilpotent if there is a flag of subbundles

$$0 = L_0 \subset L_1 \subset \cdots \subset L_k = L$$

with $\nabla(L_i) \subset L_{i-1} \otimes \Omega_X(\mathfrak{D}) \otimes_{\mathbb{C}} E$.

Let $\overline{\mathcal{N}}^{ss} \subset \overline{\mathcal{N}}$ be the open substack of semistable $\varepsilon$-connections.

Also, let $\overline{\mathcal{N}}^{ss,nn} \subset \overline{\mathcal{N}}^{ss}$ be the open substack of semistable $\varepsilon$-connections that are not nilpotent.

Taking $E = \mathbb{C}$, $\varepsilon = 1$, we see that connections are particular cases of $\varepsilon$-connections. Moreover if $\varepsilon \neq 0$, there is a unique isomorphism $E \to \mathbb{C}$ such that $\varepsilon \mapsto 1$. It follows that the open substack of $\overline{\mathcal{N}}$ corresponding to $\varepsilon$-connections with $\varepsilon \neq 0$ parameterizes all connections $(L, \nabla)$, where $L$ has rank $r$ and degree $d$, $\nabla$ has poles bounded by $\mathfrak{D}$. Thus this substack can be identified with $\mathcal{N}$.

We use the theory of good moduli spaces developed by J. Alper (see [Alp]). By definition, a quasi-compact map $p : \mathcal{S} \to S$ from a stack $\mathcal{S}$ to an algebraic space $S$ is a *good moduli space* if the direct image functor $p_*$ is exact on quasi-coherent sheaves and $p_* O_{\mathcal{S}} = O_S$. In particular, this notion reduces to the notion of quotient in the sense of geometric invariant theory when $\mathcal{S}$ is the quotient stack of a scheme by an action of an algebraic group (see [Alp, Theorem 13.6]). Note that by [Alp, Theorem 4.16(vi)] $p$ is universal among maps to schemes.

**Theorem 6** (cf. Theorem 11.3 of [Sim3])**.** (*a*) *There is a good moduli space* (*in the sense of* [Alp]) $p : \overline{\mathcal{N}}^{ss,nn} \to \overline{N}$ *such that* $\overline{N}$ *is a complete scheme.*
(*b*) *Set* $\mathcal{N}^{ss} = \overline{\mathcal{N}}^{ss} \cap \mathcal{N} = \overline{\mathcal{N}}^{ss,nn} \cap \mathcal{N}$*, then* $\mathcal{N}^{ss}$ *has a good moduli space* $N$*, which is an open subscheme of* $\overline{N}$ (*so* $N$ *is the moduli space of semistable bundles with connections*).

*Remark* 3.3. Part (b) of the theorem is clear from the construction of $\overline{N}$ in §3.2 but it also easily follows from the first claim and Corollary 3.9. Indeed, let $\mathcal{E}$ be the line bundle on $\overline{\mathcal{N}}^{ss,nn}$ whose fiber over $(L, \nabla; \varepsilon \in E)$ is $E$. As shown in

Corollary 3.9, there is a line bundle $\mathcal{E}'$ on $\overline{N}$ such that $p^*\mathcal{E}' = \mathcal{E}^{\otimes r!}$. Then by [Alp, Proposition 4.5] we can identify $p_*\mathcal{E}^{\otimes r!}$ with $\mathcal{E}'$. Thus $\varepsilon^{\otimes r!}$ gives a section of $\mathcal{E}'$; let $N$ be the complement of its zero locus. Clearly $N$ is open and $\mathcal{N} = p^{-1}(N)$. This implies the claim.

3.2. **Construction of $\overline{N}$.** The moduli of $\varepsilon$-connections (on arbitrary projective variety) is constructed by C. Simpson ([Sim1, Sim2, Sim3]); it is easy to see that his argument works in the case of $\varepsilon$-connections with singularities on curves. Let us quickly recall the construction.

Fix $\varepsilon \in \mathbb{C}$, and denote by $\mathcal{N}_\varepsilon$ the moduli stack of $\varepsilon$-connections of the form $(L, \nabla; \varepsilon \in \mathbb{C})$. Equivalently, $\mathcal{N}_\varepsilon$ is a fiber of the map

$$\overline{\mathcal{N}} \to \mathbb{A}^1/\mathbf{G_m} : (L, \nabla; \varepsilon \in E) \mapsto (\varepsilon \in E).$$

Here we identify the quotient stack $\mathbb{A}^1/\mathbf{G_m}$ with the moduli stack of pairs $(\varepsilon \in E)$, where $E$ is a one-dimensional vector space.

As $\varepsilon$ varies, the stacks $\mathcal{N}_\varepsilon$ form a family $\mathcal{N}_\bullet \to \mathbb{A}^1$ (whose fiber over $\varepsilon \in \mathbb{A}^1$ is $\mathcal{N}_\varepsilon$). The total space $\mathcal{N}_\bullet$ carries an action of $\mathbf{G_m}$ via

$$t \cdot (L, \nabla; \varepsilon \in \mathbb{C}) = (L, t\nabla; t\varepsilon \in \mathbb{C}), \quad (L, \nabla; \varepsilon \in \mathbb{C}) \in \mathcal{N}_\bullet,\ t \in \mathbf{G_m}.$$

We can identify $\overline{\mathcal{N}}$ with the quotient stack $\mathcal{N}_\bullet/\mathbf{G_m}$.

Denote by $\mathcal{N}_\varepsilon^{ss} \subset \mathcal{N}_\varepsilon$ (resp. $\mathcal{N}_\bullet^{ss} \subset \mathcal{N}_\bullet$) the open substacks of semistable $\varepsilon$-connections.

**Proposition 3.4.** (*a*) *There exists a good moduli space $\mathcal{N}_\varepsilon^{ss} \to N_\varepsilon$;*
(*b*) *As $\varepsilon \in \mathbb{C}$ varies, the spaces $N_\varepsilon$ form a family $N_\bullet \to \mathbb{A}^1$ whose fiber over $\varepsilon \in \mathbb{A}^1$ is $N_\varepsilon$. There exists a good moduli space $\mathcal{N}_\bullet^{ss} \to N_\bullet$.*

*Proof.* (a) This is a particular case of [Sim2, Theorem 4.10]. Note that [Sim2, Theorem 4.10] applies to the moduli space of semistable modules over a sheaf of split almost polynomial rings of differential operators $\Lambda$ (see [Sim2, §2], p. 77, 81 for definition). In our case, $\Lambda$ is the universal enveloping of the Lie algebroid

$$\Lambda^{\le 1} = (O_X \oplus \mathcal{T}_X(-\mathfrak{D}), [\cdot \overset{\varepsilon}{,} \cdot], \rho),$$

where $\rho$ is the composition of the natural inclusion $\mathcal{T}_X(-\mathfrak{D})$ into $\mathcal{T}_X$ with multiplication by $\varepsilon$, and

$$[f_1 + \tau_1 \overset{\varepsilon}{,} f_2 + \tau_2] = \varepsilon(\tau_1(f_2) - \tau_2(f_1) + [\tau_1, \tau_2]).$$

(b) Consider the sheaf $p_1^*\Lambda$ on $X \times \mathbb{A}^1$, where $\Lambda$ is the sheaf from (a) with $\varepsilon = 1$. Let $\Lambda^R$ be its subsheaf generated by the operators of the form $\sum \varepsilon^i \lambda_i$, where $\lambda_i \in \Lambda$ has order at most $i$, $\varepsilon$ is the coordinate on $\mathbb{A}^1$. The family $N^\bullet$ is constructed by applying Theorem 4.10 to $\Lambda^R$ relative to the projection $X \times \mathbb{A}^1 \to \mathbb{A}^1$. See [Sim2, Section on $\tau$-connections, p. 87]. $\square$

The action of $\mathbf{G_m}$ on $\mathcal{N}_\bullet$ induces its action on $N_\bullet$. (In fact an action of an algebraic group on a stack always induces an action on the good moduli space; for the proof, use the universal property of good moduli spaces and [Alp, Proposition 4.7(i)].) In particular, a point $z \in N_\bullet$ yields a morphism $\mathbf{G_m} \to N_\bullet : t \mapsto t \cdot z$. If it can be extended to a morphism from $\mathbb{A}^1 \supset \mathbf{G_m}$ (resp. from $\mathbb{P}^1 - \{0\} \supset \mathbf{G_m}$), we say that the limit $\lim_{t\to 0} t \cdot z$ (resp. $\lim_{t\to\infty} t \cdot z$) exists.

The Hitchin fibration gives the following description of $N_0$. Let

$$B := \prod_{i=1}^{r} H^0(X, \Omega_X(\mathfrak{D})^{\otimes i})$$

be the base of Hitchin fibration. Recall that the Hitchin map sends a Higgs bundle $(L, \nabla; 0 \in \mathbb{C})$ to $(c_1(\nabla), \dots, c_r(\nabla))$, where $c_i(\nabla)$ are coefficients of the characteristic polynomial of $\nabla$. Thus we get a map $\mathcal{N}_0 \to B$. This map descends to a map

$$(c_1, c_2, \dots, c_r) : N_0 \to B.$$

Denote by $N_\bullet^n \subset N_0$ the zero fiber of Hitchin fibration.

**Lemma 3.5.** *Let $z \in N_\bullet$ correspond to $(L, \nabla; \varepsilon \in \mathbb{C}) \in \mathcal{N}_\bullet^{ss}$. Then $\lim_{t\to\infty} t \cdot z$ exists if and only if $(L, \nabla; \varepsilon \in \mathbb{C})$ is nilpotent.*

*Proof.* Let us start with the 'only if' direction. If the limit exists, then $\lim_{t\to\infty} t\varepsilon$ exists, so $\varepsilon = 0$. Also, the coefficients of the characteristic polynomial of $t\nabla$ are equal to $t^i c_i(\nabla)$, and so the limit $\lim_{t\to\infty} t^i c_i(\nabla)$ exists. Therefore, $c_i(\nabla) = 0$; in other words, $\nabla$ is nilpotent.

To prove the 'if' direction, it suffices to notice that $N_\bullet^n$ is complete. Indeed, it is the zero fiber of the Hitchin map. This map is proper, the proof is similar to ([Sim2], Theorem 6.11). More precisely, one has to repeat the proof of that theorem, changing $T^*$ to $T^*(\mathfrak{D})$. $\square$

**Proposition 3.6.** *The geometric quotient $(N_\bullet - N_\bullet^n)/\mathbf{G_m}$ exists; the quotient is a complete scheme of finite type, the natural projection $N_\bullet - N_\bullet^n \to (N_\bullet - N_\bullet^n)/\mathbf{G_m}$ is an affine map.*

*Proof.* This follows from [Sim3, Theorem 11.2] and the previous lemma. Indeed, the fixed point set is closed in $N_\bullet^n$ and thus complete. The fact that $\lim_{t\to 0} t \cdot z$ exists for all $z \in N_\bullet$ follows from [Sim3, Theorem 10.1] (see also Corollary 10.2).

To show that the map is affine, note that [Sim3, Theorem 11.2] is derived from [Sim3, Theorem 11.1], which in turn uses Proposition 1.9 in [GIT], but this proposition also claims that the map is affine. $\square$

The geometric quotient $\overline{N} = (N_\bullet - N_\bullet^n)/\mathbf{G_m}$ has the properties required in Theorem 6 because of the following

**Lemma 3.7.** *Let $p : \mathcal{S} \to S$ be a good moduli space. Let $G$ be a reductive group acting on $\mathcal{S}$. Consider the induced action on $S$ and assume that there is a geometric quotient $S//G$ such that the projection $S \to S//G$ is affine. Then the induced map $\bar{p} : \mathcal{S}/G \to S//G$ is a good moduli space (the quotient in the left hand side is the stacky one).*

*Proof.* Let us decompose $\bar{p}$ as

$$\mathcal{S}/G \xrightarrow{p'} S/G \xrightarrow{p''} S//G.$$

We just have to check that $p'_*$ and $p''_*$ are exact and take the structure sheaves to the structure sheaves. This easily follows from our assumptions. $\square$

3.3. **Projectivity of $\overline{N}$.** Let us construct an ample bundle on $\overline{N}$. Fix a point $x \in X$.

**Lemma 3.8.** *Let $\alpha$ be an automorphism of $(L, \nabla; \varepsilon \in E) \in \overline{\mathcal{N}}^{ss,nn}$.*
*$(a)$ The action of $\alpha$ on $E$ is a root of unity of degree at most $r$. In particular, $\alpha$ acts trivially on $E^{\otimes r!}$.*
*$(b)$ If $\alpha$ acts trivially on $E$, then $\alpha$ acts trivially on*

$$\det \mathrm{R}\Gamma(X, L)^{\otimes r} \otimes \det(L_x)^{\otimes(rg-d-r)}.$$

*$(c)$ The automorphism $\alpha$ acts trivially on*

$$\left(\det \mathrm{R}\Gamma(X, L)^{\otimes r} \otimes \det(L_x)^{\otimes(rg-d-r)}\right)^{\otimes r!}.$$

*Proof.* (a) Note that $\alpha(\varepsilon) = \varepsilon$, so if $\varepsilon \neq 0$, then $\alpha$ acts trivially on $E$. If $\varepsilon = 0$, consider the coefficients of the characteristic polynomial of $\nabla$

$$c_i \in H^0(X, \Omega_X(\mathfrak{D})^{\otimes i}) \otimes E^{\otimes i}.$$

Since $\nabla$ is not nilpotent, there is $i$ such that $c_i \neq 0$. Now it suffices to note that $\alpha(c_i) = c_i$.
(b) We can decompose $L = \oplus_{\lambda \in \mathbb{C}} L^\lambda$, where $\alpha - \lambda$ is nilpotent on $L^\lambda$ (almost all of the summands vanish). Since $\alpha$ acts trivially on $E$, we have $\nabla(L^\lambda) \subset L^\lambda \otimes \Omega_X(\mathfrak{D}) \otimes_{\mathbb{C}} E$. By semistability of $L$, $\deg L^\lambda = \frac{d}{r} \operatorname{rk} L^\lambda$. We can then identify

$$\det \mathrm{R}\Gamma(X, L) \simeq \bigotimes \det \mathrm{R}\Gamma(X, L^\lambda)$$

and $\det L_x \simeq \bigotimes \det((L^\lambda)_x)$. Finally, $\alpha$ acts as $\lambda^{\deg L^\lambda - (g-1) \operatorname{rk} L^\lambda}$ (here $g$ is the genus of $X$) on $\det \mathrm{R}\Gamma(X, L^\lambda)$ and as $\lambda^{\operatorname{rk} L_\lambda}$ on $\det((L^\lambda)_x)$.
(c) Follows from (b) applied to $\alpha^{r!}$. □

Let us denote by $\delta$ (resp. $\mathcal{E}$, resp. $\delta_x$) the line bundle on $\overline{\mathcal{N}}$ whose fiber over $(L, \nabla; \varepsilon \in E)$ equals $\det \mathrm{R}\Gamma(X, L)$ (resp. $E$, resp. $\det(L_x)$).

**Corollary 3.9.** *The line bundles*

$$\mathcal{E}^{\otimes r!}|_{\overline{\mathcal{N}}^{ss,nn}} \quad \textit{and} \quad \left(\delta^{\otimes r} \otimes \delta_x^{\otimes(rg-d-r)}\right)^{\otimes r!}\Big|_{\overline{\mathcal{N}}^{ss,nn}}$$

*are pullbacks of line bundles on $\overline{N}$.*

*Proof.* By Lemma 3.8, automorphisms of any closed point of $\overline{\mathcal{N}}^{ss,nn}$ act trivially on the fibers of these two bundles. The statement now follows from [Alp, Theorem 10.3]. □

Denote the corresponding line bundles on $\overline{N}$ by $\mathcal{E}'$ and $\delta'$.

**Theorem 7.** *The line bundle $(\delta')^{-1} \otimes (\mathcal{E}')^{\otimes k}$ is ample on $\overline{N}$ for $k \gg 0$.*

*Proof.* Recall that by construction, $(\delta')^{-1}$ is ample on $N$, cf. Remark before [Sim2, Theorem 4.10]. Let $N_H$ be $\overline{N} - N$ with reduced scheme structure. By construction, $N_H$ is the reduction of the quotient scheme $(N_0 - N_\bullet^n)/\mathbf{G_m}$. The Hitchin map therefore induces a morphism from $N_H$ to the quotient scheme $(B - \{0\})/\mathbf{G_m}$, which is a weighted projective space. Recall [EGAII, Definition 4.6.1] that a sheaf $\mathcal{F}$ on $Y$ is called relatively ample for a map $f : Y \to Y'$ if for some cover $Y' = \cup Y'_\alpha$ with affine $Y'_\alpha$, the sheaves $\mathcal{F}|_{f^{-1}(Y'_\alpha)}$ are ample.

We make the following observations.

(a) $(\delta')^{-1}$ is relatively ample for the Hitchin map $N_0 \to B$ (since $(\delta')^{-1}$ is ample on $N_0$);

(b) $(\delta')^{-1}$ is relatively ample for the equivariant Hitchin map $N_H \to (B - \{0\})/\mathbf{G_m}$. Indeed, the relative ampleness can be proved fiberwise, cf. [EGAIII, Theorem 4.7.1] (We are thankful to Brian Conrad for the reference). On the other hand, a fiber of the equivariant Hitchin map is the categorical quotient of the corresponding fiber of the Hitchin map by the finite stabilizer of the corresponding point in $(B - \{0\})/\mathbf{G_m}$. It remains to use the following general fact: *the descent of an equivariant ample line bundle to the quotient by a finite group is ample.* This follows from [GIT, Proposition 1.15, Theorem 1.10];

(c) $\mathcal{E}'|_{N_H}$ is naturally a pullback of a sheaf on $(B - \{0\})/\mathbf{G_m}$, which we also denote by $\mathcal{E}'$;

(d) $\mathcal{E}'$ is very ample on $(B - \{0\})/\mathbf{G_m}$.

Recall from Remark 3.3 that $\varepsilon^{\otimes r!}$ yields a section $\varepsilon' \in H^0(\overline{N}, \mathcal{E}')$, whose set-theoretic zero locus is $N_H$. Denote by $N'_H$ the scheme-theoretic zero locus of $\varepsilon'$. It is a non-reduced 'thickening' of $N_H$.

*Step 1.* For integers $l, k$, consider the line bundle

$$\mathcal{L} = \mathcal{L}_{l,k} := (\delta')^{\otimes -l} \otimes \mathcal{E}'^{\otimes k}.$$

There exists $l_0$ such that for all $l > l_0$, there is $k_0 = k_0(l)$ such that for all $k > k_0$ the line bundle $\mathcal{L}|_{N_H}$ is very ample on $N_H$. This follows from [EGAII, Proposition 4.6.11] and [EGAII, Proposition 4.4.10(ii)].

*Step 2.* For any coherent sheaf $\mathcal{F}$ on $N_H$, there exists $l_0$ such that for all $l > l_0$, there is $k_0 = k_0(l)$ such that for all $k > k_0$ and all $i > 0$,

$$H^i(N_H, \mathcal{F} \otimes \mathcal{L}) = 0$$

and $\mathcal{F} \otimes \mathcal{L}$ is generated by global sections. This follows from the fact that the derived functor of global sections on $N_H$ is the composition of the derived direct image to $(B - \{0\})/\mathbf{G_m}$ and the derived functor of global sections on $(B - \{0\})/\mathbf{G_m}$.

*Step* $2'$. Same statement as in Step 2 is true with $N_H$ changed to $N'_H$. For the proof, consider a filtration of $\mathcal{F}$ with factors supported scheme-theoretically on $N_H$ and use the long exact sequence of cohomology.

*Step* $1'$. Same statement as in Step 1 is true with $N_H$ changed to $N'_H$. Indeed, set $\mathcal{F}_i := O_{N'_H}(-iN_H)$. By Step $2'$ we can assume that $\mathcal{F}_1 \otimes \mathcal{L}$ is generated by global sections and $H^0(N'_H, \mathcal{L})$ surjects onto $H^0(N_H, \mathcal{L})$. By Step 1 we can assume that $\mathcal{L}|_{N_H}$ is very ample. Let us show that $\mathcal{L}$ is very ample on $N'_H$.

We are going to use [Har, Proposition II.7.2]. Take $s \in H^0(N'_H, \mathcal{L})$ and let $N'_s$ be the open subset of $N'_H$ defined by $s \neq 0$. Then $N_s := N'_s \cap N_H$ is affine. Therefore, $N'_s$ is also affine. It suffices to show that the set $\{s'/s |\, s' \in H^0(N'_H, \mathcal{L})\}$ generates the ring $A_i := H^0(N'_s, O_{N'_H}/\mathcal{F}_i)$ for all $i$. We proceed by induction. For $i = 1$ this follows from very ampleness of $\mathcal{L}|_{N_H}$. Take $t \in A_i$; using our statement with $i = 1$, we can assume that $t \in \mathcal{F}_1/\mathcal{F}_i$. By assumption it can be written as $\sum \lambda_j(s_j/s)$, where $s_j \in H^0(N'_H, \mathcal{F}_1 \otimes \mathcal{L})$, $\lambda_j \in A_{i-1}$. It remains to use the inductive hypothesis.

For $i \gg 0$ we have $\mathcal{F}_i = 0$ and we are done.

*Step* 3. From now on, fix $l$ satisfying the conditions of Steps $1'$ and $2'$, and also such that $\mathcal{L}|_N = (\delta')^{\otimes -l}|_N$ is very ample. Then the restriction map $H^0(\overline{N}, \mathcal{L}) \to H^0(N'_H, \mathcal{L})$ is surjective for $k \gg 0$.

Indeed, this map fits into the exact sequence

$$H^0(\overline{N},\mathcal{L}) \to H^0(N'_H,\mathcal{L}) \to H^1(\overline{N},\mathcal{L}(-N'_H)) \to H^1(\overline{N},\mathcal{L}) \to H^1(N'_H,\mathcal{L})$$

According to Step $2'$, for $k \gg 0$, the rightmost term vanishes, and so the map $H^1(\overline{N},\mathcal{L}(-N'_H)) \to H^1(\overline{N},\mathcal{L})$ is surjective. On the other hand, $\mathcal{L}(-N'_H)$ equals $\mathcal{L}_{l,k-1}$, so $\dim H^1(\overline{N},\mathcal{L})$ decreases as a function of $k$ for $k \gg 0$. This dimension is finite by properness of $\overline{N}$, and therefore stabilizes. In other words, for $k \gg 0$, the map $H^1(\overline{N},\mathcal{L}(-N'_H)) \to H^1(\overline{N},\mathcal{L})$ is an isomorphism.

*Step* 4. $\mathcal{L} = \mathcal{L}_{l,k}$ is very ample on $\overline{N}$ for $k \gg 0$ (the choice of $l$ is as in Step 3).

Recall that $\mathbb{P}(V)$ denotes the projective space of hyperplanes in a vector space $V$. Choose a finite-dimensional vector space $V \subset H^0(N,\mathcal{L})$ that defines an embedding $N \hookrightarrow \mathbb{P}(V)$, and for every $k \gg 0$ a subspace $W_k \subset H^0(N'_H,\mathcal{L})$ that defines an embedding $N'_H \hookrightarrow \mathbb{P}(W_k)$. For $k \gg 0$, the space $V$ is contained in $H^0(\overline{N},\mathcal{L}\otimes(\mathcal{E}')^{-1}) \subset H^0(N,\mathcal{L})$ (because $H^0(N,\mathcal{L})$ is the limit of spaces $H^0(\overline{N},\mathcal{L})$ as $k\to\infty$), and $W_k$ can be lifted to a finite-dimensional subspace of $H^0(\overline{N},\mathcal{L})$ (which we still denote by $W_k$) by Step 3. It follows from [Har, Proposition II.7.3] that $V\varepsilon' + W_k + W_{k-1}\varepsilon'$ defines an embedding $\overline{N} \hookrightarrow \mathbb{P}(V\varepsilon' + W_k + W_{k-1}\varepsilon')$ for $k \gg 0$. Note that this proposition is stated for projective schemes only but it is valid for any proper scheme. Indeed, the projectivity is needed only for applying [Har, Corollary 5.20], but the corollary is well-known to be true with the weaker assumption. □

## 4. Properties of $\mathcal{M}$ and of its compactification

In this section we prove Theorem 1. We also prove

**Proposition 4.1.** *Let $\mathcal{F}$ be any quasi-coherent sheaf on $\mathcal{M}$. Then $H^i(\mathcal{M},\mathcal{F}) = 0$ for $i \geq 2$.*

We construct a compactification $\overline{\mathcal{M}} = \mathcal{M} \sqcup \mathcal{M}_H$ (see Proposition 4.7). We prove that the stacks $\overline{\mathcal{M}}$ form a flat family as $\mathfrak{D}$ and the local invariants of connections vary (Proposition 4.14). We give an explicit description of parabolic bundles underlying bundles with connections (Proposition 4.10). We begin with general statements but starting from Lemma 4.6 we assume that $X = \mathbb{P}^1$ and $\deg\mathfrak{D} = 4$. This assumption continues through the end of the paper.

### 4.1. Connections compatible with parabolic structure.

We start by describing parabolic bundles that possess compatible connections.

Let $X$ be a smooth projective curve of genus $g$. Let $L$ be a vector bundle of degree $d$ on $X$. Denote by $b(L) \in H^1(X,\mathcal{E}nd(L)\otimes\Omega_X)$ the class of the Atiyah sequence

$$0 \to L\otimes\Omega_X \to B(L) \to L \to 0.$$

We have the Serre duality pairing $\langle\cdot,b(L)\rangle : \mathrm{End}(L) \to \mathbb{C}$. Recall from [Ati, Proposition 18] that

$$\begin{aligned}\langle A,b(L)\rangle &= 0 \text{ if } A \text{ is nilpotent},\\ \langle \mathrm{id}_L,b(L)\rangle &= -d.\end{aligned}\tag{4.1}$$

*Remark* 4.2. (a) To match [Ati], the Serre duality pairing should include the factor of $2\pi\sqrt{-1}$.

(b) For every $A \in \mathrm{End}(L)$ the Serre duality pairing $\langle A,b(L)\rangle$ is given by

$$\langle A,b(L)\rangle = -\operatorname{tr}(A|R\Gamma(X,L)) + \operatorname{tr}(A)\chi(O_X).$$

Here $\operatorname{tr}(A|R\Gamma(X,L))$ is the alternating sum of the traces of maps on $H^i(X,L)$ induced by $A$, and $\chi(O_X) = 1 - g$. This follows from (4.1). Indeed, we can decompose $L$ into a direct sum such that the semisimple part of $A$ is scalar on each summand.

Fix $\varepsilon \in \mathbb{C}$, distinct points $x_1, \dots, x_k \in X$, and principal parts

$$A_i \in \mathcal{E}nd(L)(\infty \cdot x_i)/\mathcal{E}nd(L)$$

for the vector bundle $\mathcal{E}nd(L)$ at $x_i$ for all $i = 1, \dots, k$. To these data, we associate the sheaf of $\varepsilon$-connections $\mathcal{C} = \mathcal{C}(L, \varepsilon, \{A_i\}_{i=1}^k)$: its sections over an open subset $U \subset X$ are $\varepsilon$-connections

$$\nabla : L|_U \to L|_U \otimes \Omega_U \left(\sum_{i=1}^k \infty \cdot x_i\right)$$

such that $\nabla - A_i$ is regular at $x_i$ for all $x_i \in U$. Since $\mathcal{C}$ is a torsor over $\mathcal{E}nd(L) \otimes \Omega_X$, its isomorphism class is given by an element

$$c = c\,(L, \varepsilon, \{A_i\}_{i=1}^k) \in H^1(X, \mathcal{E}nd(L) \otimes \Omega_X).$$

**Lemma 4.3.** *For every $A \in \operatorname{End}(L)$,*

$$\langle A, c\rangle = \varepsilon\langle A, b(L)\rangle + \sum_{i=1}^k \operatorname{tr}(\operatorname{res}(A \cdot A_i)).$$

*Proof.* The torsor $\mathcal{C}$ depends linearly on the collection $(L, \varepsilon, \{A_i\}_{i=1}^k)$. We can therefore assume that either all $A_i = 0$ (and then $c = \varepsilon b(L)$) or $\varepsilon = 0$ (this case follows from the definition of Serre pairing). $\square$

It is easy to adapt Lemma 4.3 to the settings of parabolic bundles. For simplicity, we only state it for bundles of rank two. Let us fix a divisor $\mathfrak{D} = \sum n_i x_i \geq 0$ on $X$. Suppose that $L$ is a rank two vector bundle on $X$, and $\eta$ is a level $\mathfrak{D}$ parabolic structure on $L$, that is, a line subbundle $\eta \subset L|_{\mathfrak{D}}$ (cf. Definition 2.4).

Denote by $\mathcal{E}nd(L, \eta)$ the locally free sheaf of endomorphisms of $L$, preserving $\eta$; let $\operatorname{End}(L, \eta) := H^0(\mathbb{P}^1, \mathcal{E}nd(L, \eta))$ be the corresponding ring of endomorphisms.

**Corollary 4.4.** *Fix $\varepsilon \in \mathbb{C}$ and principal parts $\alpha^+, \alpha^- \in \Omega_X(\mathfrak{D})/\Omega_X$. The following conditions are equivalent:*
*(a) There exists an $\varepsilon$-connection $\nabla : L \to L \otimes \Omega_X(\mathfrak{D})$ whose 'polar part' $L|_{\mathfrak{D}} \to (L \otimes \Omega_X(\mathfrak{D}))|_{\mathfrak{D}}$ equals to $\alpha^+$ on $\eta$ and induces $\alpha^-$ on $(L|_{\mathfrak{D}})/\eta$.*
*(b) For any endomorphism $A \in \operatorname{End}(L, \eta)$, we have*

$$\operatorname{res}(A_+\alpha_+) + \operatorname{res}(A_-\alpha_-) + \varepsilon\langle A, b(L)\rangle = 0.$$

*Here $A_+, A_- \in \mathbb{C}[\mathfrak{D}]$ are the (scalar) operators induced by $A$ on $\eta$ and on $(L|_{\mathfrak{D}})/\eta$ respectively, and the residue functional $\operatorname{res} : \Omega_X(\mathfrak{D})/\Omega_X \to \mathbb{C}$ is given by*

$$\operatorname{res}\omega := \sum_{x \in \mathfrak{D}} \operatorname{res}_x \omega.$$

*Proof.* Denote by $\mathcal{H}iggs(L, \eta)$ the sheaf of Higgs fields $B : L \to L \otimes \Omega_X(\mathfrak{D})$ whose polar part $L|_{\mathfrak{D}} \to (L \otimes \Omega_X(\mathfrak{D}))|_{\mathfrak{D}}$ induces 0 on both $\eta$ and $(L|_{\mathfrak{D}})/\eta$. In other words, in any $\eta$-compatible local trivialization of $L|_{n_i x_i}$ we have

$$B = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} + \text{non-singular terms.}$$

Note that $\mathcal{H}iggs(L,\eta) \simeq \mathcal{E}nd(L,\eta)^\vee \otimes \Omega_X$. Indeed, the pairing is given by the trace of the product, and one checks in local coordinates that it is perfect.

The sheaf of connections satisfying (a) forms a torsor over $\mathcal{H}iggs(L,\eta)$; clearly, this torsor is induced from the $\mathcal{E}nd(L) \otimes \Omega_X$-torsor $\mathcal{C}(L,\varepsilon,\{A_i\}_{i=1}^n)$ for a choice of polar parts $A_i$ compatible with $\eta$ and $\alpha^\pm$. Now the claim follows from Lemma 4.3. $\square$

*Remark* 4.5. If $A_i$ has a pole of first order for all $i$, then Corollary 4.4 becomes Theorem 7.2 in [CB], which is a special case of Mihai's results [Mih1, Mih2].

4.2. **Local invariants of connections, revisited.** Let $\mathcal{N} = \mathcal{N}(X, 2, d, \mathfrak{D})$ be as in §3. Let $(L, \nabla; \varepsilon \in E) \in \overline{\mathcal{N}}$ be an $\varepsilon$-connection, where $L$ has rank 2. Let $D$ be the formal disc centered at $x_i$. Trivializing $L|_D$, we can write

$$\nabla|_D = \varepsilon \mathbf{d} + a, \qquad a \in \mathfrak{gl}(2) \otimes \Omega_D(n_i x_i) \otimes_{\mathbb{C}} E.$$

It is easy to see that $\operatorname{tr} a$ and $\det a$ are well defined (that is independent of the trivialization) as sections of $E \otimes_{\mathbb{C}} (\Omega_X(n_i x_i)/\Omega_X)$ and $E^{\otimes 2} \otimes_{\mathbb{C}} (\Omega_X^{\otimes 2}(2n_i x_i)/\Omega_X^{\otimes 2}(n_i x_i))$ respectively. Performing this operation at every $x_i$, we get well-defined sections of $E \otimes_{\mathbb{C}} (\Omega_X(\mathfrak{D})/\Omega_X)$ and $E^{\otimes 2} \otimes_{\mathbb{C}} (\Omega_X^{\otimes 2}(2\mathfrak{D})/\Omega_X^{\otimes 2}(\mathfrak{D}))$, which we denote $[\operatorname{tr} \nabla]$ and $[\det \nabla]$ respectively.

Clearly, in the case of a non-resonant connection $(L, \nabla; 1 \in \mathbb{C})$ we get

$$[\operatorname{tr} \nabla] = \nu_1 := (\alpha_i^+ + \alpha_i^-), \qquad [\det \nabla] = \nu_2 := (\alpha_i^+ \alpha_i^-),$$

where $(\alpha_i^\pm)$ is the formal type of the connection (cf. §2.1). Thus in this case the data $\big([\operatorname{tr} \nabla], [\det \nabla]\big)$ is equivalent to the formal type.

Fix $\nu_1 \in \Omega_X(\mathfrak{D})/\Omega_X$ and $\nu_2 \in \Omega_X^{\otimes 2}(2\mathfrak{D})/\Omega_X^{\otimes 2}(\mathfrak{D})$ and denote by $\overline{\mathcal{M}}$ the closed substack of $\overline{\mathcal{N}}^{ss,nn}$ parameterizing $\varepsilon$-connections such that

$$[\operatorname{tr} \nabla] = \varepsilon \otimes \nu_1, \qquad [\det \nabla] = \varepsilon^{\otimes 2} \otimes \nu_2. \tag{4.2}$$

Assume now that $X = \mathbb{P}^1$, $\deg \mathfrak{D} = 4$, $d$ is odd. Recall that in §2.1 we defined a moduli stack $\mathcal{M}$.

**Lemma 4.6.** *Every connection $(L, \nabla) \in \mathcal{M}$ is irreducible.*

*Proof.* Assume for a contradiction that $L' \subset L$ is a $\nabla$-invariant line subbundle. One checks that $\operatorname{res}_{x_i}(\nabla|_{L'}) = \operatorname{res} \alpha_i^\pm$, where $\alpha_i^\pm$ are defined in §2.1. This contradicts condition (c) of §2.1 (cf. [AL, Proposition 1]). $\square$

In particular every $(L, \nabla) \in \mathcal{M}$ is semistable, and we see that the open substack of $\overline{\mathcal{M}}$ given by $\varepsilon \neq 0$ is identified with $\mathcal{M}$. Let $\mathcal{M}_H$ be the closed substack of $\overline{\mathcal{M}}$ defined by $\varepsilon = 0$. Then $\mathcal{M} = \overline{\mathcal{M}} - \mathcal{M}_H$. By Theorem 6 and [Alp, Lemma 4.14], $\overline{\mathcal{M}}$ has a good moduli space $\overline{M}$. It follows from Theorem 7 that $\overline{M}$ is projective.

Note that $\mathcal{M}$ is a closed substack of $\mathcal{N}^{ss}$, so, using again [Alp, Lemma 4.14], we see that $M := N \cap \overline{M}$ is the good moduli space of $\mathcal{M}$. Clearly, $M$ is open in $\overline{M}$.

**Proposition 4.7.** *$\mathcal{M}_H \subset \overline{\mathcal{M}}$ is a Cartier divisor.*

This will be proved below after we prove some properties of $\mathcal{M}$.

4.3. **An affine bundle structure on $\mathcal{M}$.** Denote by $\mathcal{B}un(d) = \mathcal{B}un(\mathbb{P}^1, 2, d, \mathfrak{D})$ the moduli stack of level-$\mathfrak{D}$ parabolic bundles $(L, \eta)$ such that $L$ has degree $d$ and $\operatorname{End}(L, \eta) = \mathbb{C}$. By semicontinuity $\mathcal{B}un(d)$ is an open substack in $\overline{\mathcal{B}un}(d)$.

Consider $(L, \nabla) \in \mathcal{M}$. The formal classification (2.1) of connections shows that there is a unique level-$\mathfrak{D}$ parabolic structure $\eta$ compatible with $\nabla$ in the following sense: for any $i$ and every section $s$ of $L$ in a neighborhood of $x_i$ such that $s|_{n_i x_i} \in H^0((n_i x_i), \eta)$ we have that $\nabla s - \alpha_i^+ s$ is regular at $x_i$.

**Proposition 4.8.** *(a) If $(L, \nabla) \in \mathcal{M}$, then $L \simeq O_{\mathbb{P}^1}(m) \oplus O_{\mathbb{P}^1}(n)$ with $m - n = 1$.*
*(b) Assume that $(L, \eta)$ is the parabolic bundle corresponding to $(L, \nabla) \in \mathcal{M}$, then $\operatorname{End}(L, \eta) = \mathbb{C}$.*
*(c) The resulting map $\rho : \mathcal{M} \to \mathcal{B}un(d)$ is an affine bundle of rank 1.*

*Proof.* (a) Let us write $L = O_{\mathbb{P}^1}(m) \oplus O_{\mathbb{P}^1}(n)$ with $m > n$. Assume for a contradiction that $m - n \geq 3$. Consider the map

$$\overline{\nabla} : O_{\mathbb{P}^1}(m) \hookrightarrow L \xrightarrow{\nabla} L \otimes \Omega_{\mathbb{P}^1}(\mathfrak{D}) \twoheadrightarrow O_{\mathbb{P}^1}(n) \otimes \Omega_{\mathbb{P}^1}(\mathfrak{D}) \simeq O_{\mathbb{P}^1}(n+2).$$

It is easy to see that this map is $O_{\mathbb{P}^1}$-linear, thus it is zero. It follows that $O_{\mathbb{P}^1}(m)$ is a $\nabla$-invariant subbundle in $L$, which contradicts Lemma 4.6.

(b) Assume first that there is $A \in \operatorname{End}(L, \eta)$ such that $A$ has different eigenvalues. Then in exactly the same way as in [AL, Proposition 3] we come to contradiction with condition (c) of §2.1.

Assume now that $A \in \operatorname{End}(L, \eta)$ is not scalar and it has equal eigenvalues. Then the matrix of $A$ with respect to the decomposition $L = O_{\mathbb{P}^1}(m) \oplus O_{\mathbb{P}^1}(n)$ is

$$A = \begin{pmatrix} c & f \\ 0 & c \end{pmatrix},$$

where $c$ is a constant, $f$ is a section of $O_{\mathbb{P}^1}(m-n)$, that is a polynomial of degree at most 1.

For every $i$ choose the maximal $m_i$ such that $\eta|_{m_i x_i} = O_{\mathbb{P}^1}(m)|_{m_i x_i}$. It is easy to see that $A$ preserves $\eta$ if and only if $f$ vanishes at $x_i$ at least to order $n_i - 2m_i$ for all $i$. Hence the existence of non-scalar endomorphism implies

$$\sum_i n_i - 2m_i \leq 1. \tag{4.3}$$

Consider again $\overline{\nabla} : O_{\mathbb{P}^1}(m) \to O_{\mathbb{P}^1}(n+2)$. It follows from the compatibility that $\overline{\nabla}$ has zero of order at least $m_i$ at $x_i$. Again, $\overline{\nabla} \neq 0$, so $\sum m_i \leq 1$. However this inequality together with (4.3) would imply $\deg \mathfrak{D} \leq 3$.

(c) Consider $(L, \eta) \in \mathcal{B}un(d)$. Combining part (b), the second formula in (4.1), condition (b) of §2.1, and Corollary 4.4, we see that the fiber of $\rho$ over $(L, \eta)$ is non-empty. Thus it is a torsor over $H^0(\mathbb{P}^1, \mathcal{H}iggs(L, \eta))$. Using the identification $\mathcal{H}iggs(L, \eta) = \mathcal{E}nd(L, \eta)^\vee \otimes \Omega_X$, we obtain

$$\begin{aligned} \dim H^0(\mathbb{P}^1, \mathcal{H}iggs(L, \eta)) = \dim H^1(\mathbb{P}^1, \mathcal{E}nd(L, \eta)) = 1 - \chi(\mathcal{E}nd(L, \eta)) = \\ 1 - \deg(\mathcal{E}nd(L, \eta)) - \operatorname{rk}(\mathcal{E}nd(L, \eta)) = 1. \end{aligned} \tag{4.4}$$

We see that $H^0(\mathbb{P}^1, \mathcal{H}iggs(L, \eta))$ form a vector bundle over $\mathcal{B}un(d)$, and $\mathcal{M}$ is a torsor over this bundle. □

*Remark* 4.9. Note that, contrary to the case of regular singularities, this proposition is not valid for $n > 4$ because the proof of part (b) is specific for this case.

4.4. **Parabolic bundles.** Let $(L, \eta)$ be a rank 2 parabolic bundle. We define the *lower modification* of $(L, \eta)$ at $x_i$ as the vector bundle $L_i$ whose sheaf of sections is

$$\{s \in L : s|_{n_i x_i} \in \eta|_{n_i x_i}\}.$$

Clearly, $\deg L_i = \deg L - n_i$. We shall also use the lower modification $L_\eta$ of $L$ at all $x_i$: its sheaf of sections is

$$\{s \in L : s|_{\mathfrak{D}} \in \eta\}.$$

Note that $\eta$ induces parabolic structures on $L_i$ and $L_\eta$. For example in the case of $L_\eta$ we get an exact sequence

$$0 \to \eta \otimes O_{\mathbb{P}^1}(-\mathfrak{D})|_{\mathfrak{D}} \to L(-\mathfrak{D})|_{\mathfrak{D}} \to L_\eta|_{\mathfrak{D}},$$

thus the image $\eta'$ of $L(-\mathfrak{D})|_{\mathfrak{D}}$ in $L_\eta|_{\mathfrak{D}}$ is a parabolic structure on $L_\eta$. Upon choosing local coordinates $z_i$ at each $x_i$, $\eta'$ can be identified with $(L|_{\mathfrak{D}})/\eta$. It is easy to see that $\mathrm{End}(L, \eta) = \mathrm{End}(L_\eta, \eta')$ and a similar statement is true for $L_i$ with induced parabolic structure.

Recall that $P$ is a projective line doubled at the points of the support of $\mathfrak{D}$.

**Proposition 4.10.** *$\mathcal{B}un(d) \simeq P \times B(\mathbf{G_m})$, where $B(\mathbf{G_m}) := pt/\mathbf{G_m}$ is the classifying stack of $\mathbf{G_m}$.*

*Proof.* The idea of the proof was suggested to the first author by V. Drinfeld.

*Step 1. We can assume $d = -1$.* Let us pick a point $\infty \in \mathbb{P}^1 - \mathfrak{D}$. Then the map $(L, \eta) \mapsto (L \otimes O_{\mathbb{P}^1}(\infty), \eta)$ identifies $\mathcal{B}un(\mathbb{P}^1, 2, d, \mathfrak{D})$ with $\mathcal{B}un(\mathbb{P}^1, 2, d+2, \mathfrak{D})$. Since $d$ is odd, the statement follows.

*Step 2.* $L \simeq O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-1)$. Indeed, it follows from Proposition 4.8(c) that $(L, \eta)$ corresponds to a connection $(L, \nabla) \in \mathcal{M}$, thus we can use Proposition 4.8(a).

*Step 3.* The discussion, preceding this proposition, shows that $(L, \eta) \in \mathcal{B}un(-1)$ implies $(L_\eta \otimes O_{\mathbb{P}^1}(2\infty), \eta') \in \mathcal{B}un(-1)$ and therefore $L_\eta \simeq O_{\mathbb{P}^1}(-2) \oplus O_{\mathbb{P}^1}(-3)$.

Let $\tilde{P}$ be the moduli stack of collections $(L, \eta, O_{\mathbb{P}^1} \hookrightarrow L, O_{\mathbb{P}^1}(-2) \hookrightarrow L_\eta)$, where $(L, \eta) \in \mathcal{B}un(-1)$. Note that there is a unique up to scalar map $O_{\mathbb{P}^1} \to L$ and a unique up to scalar map $O_{\mathbb{P}^1}(-2) \to L_\eta$. Thus $\tilde{P}$ is a principal $\mathbf{G_m} \times \mathbf{G_m}$-bundle on $\mathcal{B}un(-1)$.

*Step 4.* For a point of $\tilde{P}$ we get a map $\varphi : O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-2) \to L$. We claim that this map is injective. Indeed, let $m_i$ be as in the proof of Proposition 4.8(b). If the image of $\varphi$ is a line subbundle, then $\sum m_i \geq 2$. But we saw (again in the proof of Proposition 4.8(b)) that this is impossible.

Thus $\varphi$ has a simple zero at a single point $q$. Note that $\operatorname{Ker} \varphi(q)$ does not coincide with the fiber of $O_{\mathbb{P}^1}$ (because $O_{\mathbb{P}^1} \to L$ is an embedding of vector bundles). That is, the kernel of $\varphi(q)$ is spanned by $(p, 1)$, where $p$ is a point in the fiber of $O_{\mathbb{P}^1}(2)$ over $q$. (More canonically, $p$ is a homomorphism from the fiber of $O_{\mathbb{P}^1}(-2)$ to that of $O_{\mathbb{P}^1}$.) The pair $(p, q)$ completely describes $L$ as an upper modification of $O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-2)$: the sheaf of sections of $L(-q)$ is

$$\{(s_1, s_2) \in O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-2) |\, s_1(q) = p s_2(q)\}. \tag{4.5}$$

*Step 5.* Similarly, we get a map $\varphi' : O_{\mathbb{P}^1}(-\mathfrak{D}) \oplus O_{\mathbb{P}^1}(-2) \to L_\eta$. It also has exactly one simple zero. Note that $\det \varphi = \det \varphi'$ (since $\varphi$ and $\varphi'$ can be identified on $\mathbb{P}^1 - \mathfrak{D}$), so the zero is at the same point $q$. Then $\operatorname{Ker} \varphi'(q)$ is spanned by $(1, p')$, where $p'$ is in the fiber of $O_{\mathbb{P}^1}(2)$ (more properly, $p'$ is a homomorphism between the

fiber of $O_{\mathbb{P}^1}(-\mathfrak{D})$ and that of $O_{\mathbb{P}^1}(-2)$). Again, $(p', q)$ completely determines $L_\eta$, indeed, the sheaf of sections of $L_\eta(-q)$ is

$$\{(s_1, s_2) \in O_{\mathbb{P}^1}(-\mathfrak{D}) \oplus O_{\mathbb{P}^1}(-2) \,|\, p's_1(q) = s_2(q)\}. \tag{4.6}$$

*Step 6.* Note that $(p, p', q)$ determines the inclusion $L_\eta \hookrightarrow L$ uniquely as well because it determines it on $\mathbb{P}^1 - \mathfrak{D}$, thus this triple determines a point of $\tilde{P}$. We must have $L_\eta \subset L$. Looking at (4.5) and (4.6) it is easy to see that this condition is exactly $pp' = f(q)$, where $f$ is the canonical section of $O_{\mathbb{P}^1}(\mathfrak{D})$ (thus the zero locus of $f$ is exactly $\mathfrak{D}$). This makes sense: the product $pp'$ is in the fiber of $O_{\mathbb{P}^1}(\mathfrak{D})$.

Let $P'$ be the set of triples $(p, p', q)$ as above subject to the condition $pp' = f(q)$. Every such point determines a parabolic bundle $(L, \eta)$ but some of these bundles can have extra automorphisms. In other words, $\tilde{P} \subset P'$.

Clearly, $P'$ is fibered over $\mathbb{P}^1$ with coordinate $q$, and the fiber over $x$ is either a hyperbola or a cross, depending on whether $x$ is in $\mathfrak{D}$ or not.

Finally, we need to mod out the embeddings $O_{\mathbb{P}^1} \hookrightarrow L$ and $O_{\mathbb{P}^1}(-2) \hookrightarrow L_\eta$. If we scale one of them by $a$ and the other by $b$, we get

$$(p, p', q) \mapsto ((a/b)p, (b/a)p', q).$$

Therefore

- The only points with extra automorphisms are of the form $(0, 0, q)$, $q \in \mathfrak{D}$ (the centers of the crosses);
- The stable locus $\tilde{P}$ is exactly the part of $P'$ that is smooth over $\mathbb{P}^1$.
- We have $\mathcal{B}un(-1) = \tilde{P}/\mathbf{G_m}^2$. Since the diagonal group $a = b$ acts trivially, this stack is $(\tilde{P}/\mathbf{G_m}) \times B(\mathbf{G_m})$. Clearly, $\tilde{P}/\mathbf{G_m} = P$.

□

*Remark* 4.11. Note that we can (and shall) view $P$ as the moduli space of collections

$$(L, \eta, O_{\mathbb{P}^1}(-2) \hookrightarrow L_\eta).$$

*Proof of Theorem 1.* By Propositions 4.8(c) and 4.10 $\mathcal{M}$ is a smooth connected algebraic stack of dimension 1. To prove that $\mathcal{M} = M \times B(\mathbf{G_m})$ consider the moduli stack of triples $(L, \nabla, O_{\mathbb{P}^1}(-2) \hookrightarrow L_\eta)$, where $(L, \nabla) \in \mathcal{M}$. We have $\mathcal{M} = M'/\mathbf{G_m}$, where $\mathbf{G_m}$ acts by rescaling the embedding $O_{\mathbb{P}^1}(-2) \hookrightarrow L_\eta$. It is easy to see that this action is trivial, so that $\mathcal{M} = M' \times B(\mathbf{G_m})$. On the other hand, Lemma 4.6 shows that connections in $\mathcal{M}$ have only scalar automorphisms, thus $M'$ is an algebraic space. Therefore $\mathcal{M} \to M'$ is the good moduli space and $M' = M$ by uniqueness of good moduli spaces. We see that $\mathcal{M}$ is a neutral gerbe over $M$. Next, we have a cartesian diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "\rho"] \\ P \arrow[r] & \mathcal{B}un(d). \end{tikzcd}$$

Thus $M$ is an affine bundle over $P$. It follows that $M$ is a smooth surface. Finally, $M$ is quasi-projective, since it is open in $\overline{M}$. □

*Proof of Proposition 4.1.* The map $\rho : \mathcal{M} \to \mathcal{B}un(d)$ is an affine bundle, thus it is an affine morphism. On the other hand, the good moduli space of $\mathcal{B}un(d)$ is $P$, which is a 1-dimensional scheme. □

4.5. **$\overline{\mathcal{M}}$ is a locally complete intersection.**

**Lemma 4.12.** *If $(L, \nabla; \varepsilon \in E) \in \overline{\mathcal{M}}$, then $L \simeq O_{\mathbb{P}^1}(n) \oplus O_{\mathbb{P}^1}(m)$ with $m - n = 1$.*

*Proof.* For $\varepsilon \neq 0$ this is Proposition 4.8 (a). Let $\varepsilon = 0$ and assume that $m - n \geq 3$. Then the same argument as in Proposition 4.8 (a) shows that $O_{\mathbb{P}^1}(m)$ is $\nabla$-invariant, which contradicts semistability. □

**Proposition 4.13.** *$\overline{\mathcal{M}}$ is a locally complete intersection.*

*Proof.* Let $\mathfrak{D}' \supset \mathfrak{D}$ be a divisor on $X$. Consider the moduli stack $\widetilde{\mathcal{N}}(\mathfrak{D}') \subset \overline{\mathcal{N}}(\mathbb{P}^1, 2, d, \mathfrak{D}')$ parameterizing $(L, \nabla; \varepsilon \in E)$, where $L \simeq O_{\mathbb{P}^1}(n) \oplus O_{\mathbb{P}^1}(m)$ with $m - n = 1$, $m + n = d$, $\nabla : L \to L \otimes \Omega_{\mathbb{P}^1}(\mathfrak{D}') \otimes_{\mathbb{C}} E$ is an $\varepsilon$-connection, $(L, \nabla)$ is semistable and non-nilpotent. It is enough to show that if $\deg \mathfrak{D}'$ is big enough, then (i) $\widetilde{\mathcal{N}}(\mathfrak{D}')$ is smooth and (ii) $\overline{\mathcal{M}}$ is defined by $\dim \widetilde{\mathcal{N}}(\mathfrak{D}') - \dim \overline{\mathcal{M}}$ equations in $\widetilde{\mathcal{N}}(\mathfrak{D}')$ (note that $\overline{\mathcal{M}} \subset \widetilde{\mathcal{N}}(\mathfrak{D}')$ by Lemma 4.12).

For (i) it is enough to show that the map $\widetilde{\mathcal{N}}(\mathfrak{D}') \to \mathbb{A}^1/\mathbf{G_m}$ sending $(L, \nabla; \varepsilon \in E)$ to $\varepsilon \in E$ is smooth. The relative deformation complex of this map at $(L, \nabla; \varepsilon \in E)$ is

$$\mathcal{G}^\bullet := (\mathcal{E}nd(L) \xrightarrow{\operatorname{ad} \nabla} \mathcal{E}nd(L) \otimes \Omega_{\mathbb{P}^1}(\mathfrak{D}') \otimes_{\mathbb{C}} E),$$

so that the obstruction to smoothness is in $H^1(\mathbb{P}^1, \mathcal{E}nd(L) \otimes \Omega_{\mathbb{P}^1}(\mathfrak{D}') \otimes_{\mathbb{C}} E)$. This space vanishes for $\deg \mathfrak{D}'$ big enough because $L \simeq O_{\mathbb{P}^1}(n) \oplus O_{\mathbb{P}^1}(m)$, with $m - n = 1$.

For (ii) note that $\dim \widetilde{\mathcal{N}}(\mathfrak{D}') = -\chi(\mathcal{G}^\bullet) = 4 \deg \mathfrak{D}' - 8$.

In Corollary 6.7 below, we shall give an explicit description of $\mathcal{M}_H$; this description implies that $\dim \mathcal{M}_H = 0$. Combining this with Theorem 1, we see that $\dim \overline{\mathcal{M}} = 1$.

Further, let $\mathcal{L}$ be the vector bundle on $\widetilde{\mathcal{N}}(\mathfrak{D}')$ whose fiber at $(L, \nabla; \varepsilon \in E)$ is $\mathcal{E}nd(L) \otimes (O_{\mathbb{P}^1}(\mathfrak{D}')/O_{\mathbb{P}^1}(\mathfrak{D})) \otimes_{\mathbb{C}} E$. The polar part of $\nabla$ gives rise to a section of $\mathcal{L}$ and $\widetilde{\mathcal{N}}(\mathfrak{D}) \subset \widetilde{\mathcal{N}}(\mathfrak{D}')$ is given by the zero locus of this section. Thus $\widetilde{\mathcal{N}}(\mathfrak{D})$ is locally cut out by $4(\deg \mathfrak{D}' - \deg \mathfrak{D})$ equations. It follows from the definition of $\overline{\mathcal{M}}$ (cf. §4.2) that $\overline{\mathcal{M}}$ is cut out from $\widetilde{\mathcal{N}}(\mathfrak{D})$ by $2 \deg \mathfrak{D} - 1$ equations (note that the sum of residues of $\nabla$ is equal to $-d$). □

*Proof of Proposition 4.7.* $\mathcal{M}_H$ is given by $\varepsilon = 0$, so we only need to check that $\varepsilon$ is locally not a zero divisor on $\overline{\mathcal{M}}$. However, if it was the case, $\mathcal{M}_H$ would contain a component of $\overline{\mathcal{M}}$ (set-theoretically), and we would come to a contradiction with complete intersections having pure dimension. □

4.6. **Universal Moduli Spaces.** Recall that $\mathfrak{D} = \sum n_i x_i$. Fix $\infty \in \mathbb{P}^1 \setminus \mathfrak{D}$.

Consider a moduli space $\mathcal{B}$, parameterizing local invariants of connections, that is, triples $(\mathfrak{D}, \nu_1, \nu_2)$, where $\mathfrak{D}$ is a degree 4 divisor on $\mathbb{P}^1$ such that $\infty \notin \operatorname{supp} \mathfrak{D}$, $\nu_1 \in \Omega_{\mathbb{P}^1}(\mathfrak{D})/\Omega_{\mathbb{P}^1}$, $\nu_2 \in \Omega_{\mathbb{P}^1}^{\otimes 2}(2\mathfrak{D})/\Omega_{\mathbb{P}^1}^{\otimes 2}(\mathfrak{D})$, and the sum of residues of $\nu_1$ equals $-d$, cf. §4.2. We can identify such $\mathfrak{D}$ with roots of degree 4 monic polynomial $p(z)$, then we can write uniquely

$$\nu_1 = \frac{a_0 + a_1 z + a_2 z^2 + d z^3}{p(z)} \mathbf{d}z, \qquad \nu_2 = \frac{b_0 + b_1 z + b_2 z^2 + b_3 z^3}{p(z)^2} \mathbf{d}z \otimes \mathbf{d}z.$$

Here $z$ is the standard coordinate on $\mathbb{P}^1$. Thus $\mathcal{B} \simeq \mathbb{C}^{11}$. Note that the subset of points in $\mathcal{B}$ satisfying conditions of §2.1 is open in *analytic* topology.

As $(\mathfrak{D}, \nu_1, \nu_2)$ varies, we obtain a family $\overline{\mathcal{M}}_{univ} \to \mathcal{B}$ of moduli stacks. In §4.2 we fixed $\mathfrak{D}$, $\nu_1$ and $\nu_2$. Denote the corresponding point of $\mathcal{B}$ by $t_0$. Then the fiber of $\overline{\mathcal{M}}_{univ} \to \mathcal{B}$ over $t_0$ is $\overline{\mathcal{M}}$. Our goal is to prove

**Proposition 4.14.** *The family $\overline{\mathcal{M}}_{univ} \to \mathcal{B}$ is a flat family of stacks in a Zariski neighborhood of $t_0 \in \mathcal{B}$.*

*Proof.* Similarly to the previous subsection we prove that $\overline{\mathcal{M}}_{univ}$ is a locally complete intersection of dimension $\dim \mathcal{B} + 1$. It follows that the fibers of $\overline{\mathcal{M}}_{univ} \to \mathcal{B}$ are at least 1-dimensional. By semicontinuity, there is a neighborhood of $t_0$, where fibers are 1-dimensional. It remains to note, that by [EGAIV, Proposition 6.15] a morphism from a locally complete intersection to a smooth scheme with equidimensional fibers is flat. □

## 5. Generalized line bundles on generalized elliptic curves

In this section, we provide a version of the Fourier-Mukai transform for singular degenerations of elliptic curves. This generalization is not surprising, and the case of singular reduced irreducible genus one curve (nodal or cuspidal) is well known ([BK], see also [BZN, Theorem 5.2]).

5.1. **Generalized elliptic curves.** For the purposes of this paper, it is important to work with all double covers of $\mathbb{P}^1$ ramified at four points, including reducible covers (see Remark 6.4). We were unable to find a discussion of this case in the literature. However, our argument works in greater generality, and we therefore consider the following class of curves

**Definition 5.1.** A projective curve $Y$ is *generalized elliptic* if $H^0(Y, O_Y) = \mathbb{C}$ (in particular, $Y$ is connected and has no embedded points) and the dualizing sheaf of $Y$ is trivial. In particular, $Y$ is Gorenstein and has arithmetic genus 1.

*Remark* 5.2. In fact the dualizing sheaf of $Y$ is $O_Y \otimes_{\mathbb{C}} H^1(Y, O_Y)^\vee$ rather than $O_Y$. Indeed, the first cohomology group of the dualizing sheaf must be trivialized.

*Example* 5.3. Any plane cubic (reduced or not) is a generalized elliptic curve in this sense.

*Remark* 5.4. Note that we do not assume that singularities of $Y$ are planar. For example, an intersection of two space quadrics is a generalized elliptic curve, even if the two quadrics are cones with a common vertex. In this case, the intersection is a union of four lines that meet at the vertex.

Denote by $\Sigma$ the collection of generic points of $Y$ (by definition a point $s \in Y$ is generic if its local ring $O_{Y,s}$ is Artinian). For a sheaf $\ell$ on $Y$ and $s \in \Sigma$, we denote by $\operatorname{rk}_s \ell$ the length of the stalk $\ell_s$ as a module over the local ring $O_{Y,s}$. In particular, $m(s) := \operatorname{rk}_s O_Y$ is the multiplicity of the corresponding irreducible component.

Fix a weight function $w : \Sigma \to \mathbb{R}^{>0}$ and set

$$\operatorname{rk} \ell = \operatorname{rk}_w \ell := \sum_{s \in \Sigma} w(s) \operatorname{rk}_s \ell.$$

We can now use this notion of rank (and the corresponding notion of slope) to define a stability of coherent sheaves on $Y$.

**Definition 5.5.** A coherent sheaf $\ell \neq 0$ of pure dimension 1 is said to be *semistable* if for any proper subsheaf $\ell_0 \subset \ell$, $\ell_0 \neq 0, \ell$, we have

$$\frac{\chi(\ell)}{\operatorname{rk}\ell} \geq \frac{\chi(\ell_0)}{\operatorname{rk}\ell_0}.$$

If the inequality is strict, $\ell$ is *stable*.

**Definition 5.6.** We say that a sheaf $\ell$ is a *generalized line bundle* on $Y$ if it is of pure dimension 1 and its length at all generic points of $Y$ equals to the multiplicity of the corresponding component: $\operatorname{rk}_s \ell = m(s)$ for $s \in \Sigma$.

By definition, $\deg \ell := \chi(\ell) - \chi(O_Y) = \chi(\ell)$.

Denote by $\overline{\mathcal{P}ic}^{\,d}(Y)$ the stack of generalized line bundles of degree $d$ on $Y$, and let $\overline{\mathcal{P}ic}_s^{\,d}(Y) \subset \overline{\mathcal{P}ic}_{ss}^{\,d}(Y) \subset \overline{\mathcal{P}ic}^{\,d}(Y)$ stand for the open substacks of stable and semistable generalized line bundles, respectively.

Let $\mathcal{P} := O_{Y\times Y}(-\Delta)$ be the ideal sheaf of the diagonal $\Delta \subset Y \times Y$. Note that $\mathcal{P}$ is flat over each of the factors, being the kernel of a surjection of flat sheaves, therefore, $\mathcal{P}$ is a $Y$-family of degree $-1$ generalized line bundles on $Y$, so it defines a map $Y \to \overline{\mathcal{P}ic}^{\,-1}(Y)$. (Note that $\mathcal{P}$ is not in general flat over the product.) The above map naturally extends to a map

$$Y \times B(\mathbf{G_m}) \to \overline{\mathcal{P}ic}^{\,-1}(Y). \tag{5.1}$$

Explicitly, for a test scheme $S$, the map (5.1) assigns to a morphism $\psi : S \to Y$ and a line bundle $L$ on $S$ (recall that a line bundle on $S$ is the same as a map $S \to B(\mathbf{G_m})$) the sheaf $p_1^* L \otimes O_{S\times Y}(-\Gamma_\psi)$ on $S \times Y$, viewed as an $S$-family of degree $-1$ generalized line bundles on $Y$, that is, as a morphism $S \to \overline{\mathcal{P}ic}^{\,-1}(Y)$. Here $\Gamma_\psi \subset S \times Y$ is the graph of $\psi$.

Our goal is to prove the following claims

**Proposition 5.7.** *The map* (5.1) *is an isomorphism*

$$Y \times B(\mathbf{G_m}) \xrightarrow{\sim} \overline{\mathcal{P}ic}_s^{\,-1}(Y) = \overline{\mathcal{P}ic}_{ss}^{\,-1}(Y).$$

*Remark* 5.8. In particular, $\overline{\mathcal{P}ic}_s^{\,-1}(Y) = \overline{\mathcal{P}ic}_{ss}^{\,-1}(Y)$ does not depend on $w$.

**Theorem 8.** *The Fourier-Mukai transform with kernel* $\mathcal{P}$

$$\mathcal{D}^b(Y) \to \mathcal{D}^b(Y) : \mathcal{F} \mapsto Rp_{1,*}(\mathcal{P} \otimes^L p_2^* \mathcal{F})$$

*is an auto-equivalence of the category of* $\mathcal{D}^b(Y)$ (*the bounded derived category of quasi-coherent sheaves on* $Y$). *Here* $p_1, p_2 : Y \times Y \to Y$ *are projections.*

*Remark* 5.9. Proposition 5.7 allows us to identify $Y$ and the coarse moduli space of $\overline{\mathcal{P}ic}_s^{\,-1}(Y)$. In fact, the moduli space is fine: the Poincaré sheaf $\mathcal{P}$ is a universal sheaf on $Y \times Y$. Theorem 8 then provides a Fourier-Mukai transform between $Y$ and the coarse moduli space.

Also, consider the Serre dual $\mathcal{P}^\vee := \mathcal{H}om(\mathcal{P}, O_{Y\times Y})$ of $\mathcal{P}$. Then Corollary 5.11 implies that an analogue of Proposition 5.7 holds for $\mathcal{P}^\vee$: it provides an isomorphism $Y \times B(\mathbf{G_m}) \xrightarrow{\sim} \overline{\mathcal{P}ic}_s^{\,1}(Y) = \overline{\mathcal{P}ic}_{ss}^{\,1}(Y)$. The Fourier-Mukai transform given by $\mathcal{P}^\vee$ is also an equivalence (up to cohomological shift, it is inverse to that given by $\mathcal{P}$, see §5.3).

5.2. **Stable generalized line bundles.** Let us prove Proposition 5.7. We start with some remarks about duality on $Y$.

**Lemma 5.10.** *(a) Let $\ell$ be a coherent sheaf on $Y$ of pure dimension 1. Then $\ell$ is Cohen-Macaulay: $\ell^\vee := R\mathcal{H}om(\ell, O_Y) = \mathcal{H}om(\ell, O_Y)$ is a coherent sheaf of pure dimension 1. (Recall that $O_Y$ is the dualizing sheaf on $Y$.)*
*(b) Let $S$ be a locally Noetherian scheme (or a stack) and let $\ell$ be an $S$-family of coherent sheaves of pure dimension 1 on $Y$; that is, $\ell$ is a coherent sheaf on $S \times Y$ such that $\ell$ is flat over $S$ and its restriction to fibers over $s \in S$ have pure dimension 1. Then $\ell^\vee := R\mathcal{H}om(\ell, O_{S\times Y}) = \mathcal{H}om(\ell, O_{S\times Y})$ is a coherent sheaf.*
*(c) In the assumptions of part (b), $\ell^\vee$ is flat over $S$, and for any point $s \in S$, we have $(\ell^\vee)|_{\{s\}\times Y} = (\ell|_{\{s\}\times Y})^\vee$. In other words, duality respects families.*

*Proof.* We have $\mathcal{E}xt^i(\ell, O_Y) = 0$ for $i \gg 0$ (since the dualizing sheaf has finite injective dimension). Recall that a coherent sheaf with zero fibers vanishes, thus it suffices to check that the derived restriction $L\iota_y^* R\mathcal{H}om(\ell, O_Y)$ is concentrated in non-positive cohomological degrees for any closed point $\iota_y : \{y\} \hookrightarrow Y$. However, its dual is $R\iota_y^! \ell[1]$ (since Serre duality permutes $L\iota_y^*$ and $R\iota_y^!$), which is concentrated in non-negative degrees as $\ell$ is of pure dimension 1.

For part (b), let us show that for any coherent sheaf $\mathcal{F}$ on $S$, the sheaf $\mathcal{E}xt^i(\ell, p_1^*\mathcal{F})$ vanishes for $i > 0$. The statement is local in $S$, so we may assume that $S$ is an affine Noetherian scheme without loss of generality. If $S$ is a point, the claim follows from part (a).

For general $S$, the problem is that we do not know *a priori* that $\mathcal{E}xt^i$ vanishes for $i \gg 0$. To circumvent this problem we proceed by Noetherian induction on $\operatorname{supp}(\mathcal{F})$. Fix a generic point $\iota_s : \operatorname{Spec} O_{S,s} \hookrightarrow \operatorname{supp}(\mathcal{F})$, and consider the adjunction morphism

$$\mathcal{F} \to \iota_{s,*}\iota_s^*\mathcal{F}.$$

Both its kernel and its cokernel are unions of coherent sheaves supported by strictly smaller closed subsets, so the induction hypothesis implies that the induced map

$$\mathcal{E}xt^i(\ell, p_1^*\mathcal{F}) \to \mathcal{E}xt^i(\ell, p_1^*\iota_{s,*}\iota_s^*\mathcal{F})$$

is an isomorphism for $i > 1$. Also, part (a) implies that

$$\mathcal{E}xt^i(\ell, p_1^*\iota_{s,*}\iota_s^*\mathcal{F}) = 0 \qquad (i > 0),$$

so that $\mathcal{E}xt^i(\ell, p_1^*\mathcal{F}) = 0$ for $i > 1$.

Now note that for any point $\iota_s : \{s\} \hookrightarrow S$,

$$L(\iota_s \times \mathrm{id}_Y)^* R\mathcal{H}om(\ell, p_1^*\mathcal{F}) = R\mathcal{H}om(\ell|_{\{s\}\times Y}, L\iota_s^*\mathcal{F} \boxtimes O_Y) \tag{5.2}$$

is concentrated in non-positive cohomological dimensions, thus $\mathcal{E}xt^1(\ell, p_1^*\mathcal{F}) = 0$. This completes the proof of part (b).

Part (c) follows by taking $\mathcal{F} = O_S$ in (5.2). □

**Corollary 5.11.** *Let $\ell$ be a generalized line bundle on $Y$. Then so is $\ell^\vee$, and $\deg \ell^\vee = -\deg \ell$. Moreover, $\ell$ is (semi)stable if and only if $\ell^\vee$ is (semi)stable.*

*For every $d$, the duality $\ell \mapsto \ell^\vee$ defines an isomorphism $\overline{\mathcal{P}ic}^{\,d}(Y) \xrightarrow{\sim} \overline{\mathcal{P}ic}^{\,-d}(Y)$ and also isomorphisms $\overline{\mathcal{P}ic}^{\,d}_{ss}(Y) \xrightarrow{\sim} \overline{\mathcal{P}ic}^{\,-d}_{ss}(Y)$ and $\overline{\mathcal{P}ic}^{\,d}_{s}(Y) \xrightarrow{\sim} \overline{\mathcal{P}ic}^{\,-d}_{s}(Y)$.*

*Proof.* If $\ell$ is a sheaf of pure dimension 1 on $Y$, then $\ell^\vee$ is also a sheaf, and by Serre duality $\chi(\ell^\vee) = -\chi(\ell)$. The duality for modules over Artinian rings shows that $\ell$ is also a generalized line bundle. Assume that $\ell$ is a (semi)stable generalized line bundle. Let $\ell_0 \subset \ell^\vee$ be a subsheaf (necessarily of pure dimension 1). Without loss of generality we can assume that $\ell^\vee/\ell_0$ is also of pure dimension 1. Then by part (a) of the lemma we get a surjection $\ell \twoheadrightarrow \ell_0^\vee$, and we see that the (semi)stability property for $\ell$ implies the similar property for $\ell^\vee$. The remaining claims are obvious. □

**Lemma 5.12.** *$O_Y$ is stable.*

*Proof.* By definition, we need to show that $O_Y$ has no proper subsheaves $\ell_0 \subset O_Y$ with $\chi(\ell_0) \geq 0$. Assume the converse. Without loss of generality, we may assume that $\ell_1 := O_Y/\ell_0$ has pure dimension 1. By Lemma 5.10(a), its dual

$$\ell_1^\vee = \mathcal{H}om(\ell_1, O_Y) = R\mathcal{H}om(\ell_1, O_Y)$$

is a sheaf. Since $\ell_1 = O_Y/\ell_0$ has a global section (image of $1 \in O_Y$), and $\chi(\ell_1) = -\chi(\ell_0) \leq 0$, we see that $H^1(Y, \ell_1) \neq 0$. Then by Serre duality $H^0(Y, \ell_1^\vee) \neq 0$, which is impossible, because $\ell_1^\vee \subset O_Y$ is a proper subsheaf. □

Let us prove Proposition 5.7 on the level of points.

**Proposition 5.13.** *Let $\ell$ be a generalized line bundle of degree $-1$. Then $\ell$ is stable if and only there exists an isomorphism $\ell \simeq O_Y(-y)$, where $O_Y(-y) \subset O_Y$ is the ideal sheaf of a closed point $y \in Y$. The isomorphism is unique up to scaling, and the point $y$ is uniquely determined by $\ell$. Also, stability of $\ell$ is equivalent to semistability.*

*Proof.* For a sheaf $\ell$ of degree $-1$, both stability and semistability mean that all quotient sheaves of $\ell$ have non-negative Euler characteristic (excluding $\ell$ itself). In particular, $O_Y(-y)$ is stable; indeed, for any quotient $\mathcal{F}$ of $O_Y(-y)$ we have a corresponding quotient $\mathcal{F}'$ of $O_Y$ and $\chi(\mathcal{F}') = \chi(\mathcal{F})+1$, so we can use Lemma 5.12.

Conversely, suppose $\ell$ is stable. Then $H^0(Y, \ell) = 0$, because $O_Y$ is stable and its slope is greater than the slope of $\ell$. Therefore, $H^1(Y, \ell)$ is one-dimensional; by Serre duality, this gives a non-zero homomorphism $\kappa : \ell \to O_Y$, unique up to scaling. Since $\ell$ is a generalized line bundle, $\kappa$ cannot be surjective. Thus, by stability of $O_Y$, the image has negative Euler characteristic; together with stability of $\ell$, this implies that $\kappa$ is an embedding. □

*Remark* 5.14. Once it is proved that $O_Y(-y)$ is stable for all $y \in Y$, Proposition 5.13 can also be derived from Theorem 8 together with Remark 5.9. More generally, let $L$ be a semistable coherent sheaf of slope $-1$, so that $\deg L = -\operatorname{rk} L$. We claim that the Fourier-Mukai transform of $L^\vee$ is of the form $\mathcal{F}[-1]$, where $\mathcal{F}$ is a coherent sheaf of finite length that equals $\operatorname{rk} L$, assuming that the weight function $w$ is normalized so that $\operatorname{rk} O_Y = 1$ (cf. [BK, Theorem 2.21]). Note that the converse implication is obvious: if the Fourier-Mukai transform of $L^\vee$ is of this form, then $L$ is obtained by successive extension from sheaves of the form $O_Y(-y)$, which implies semistability.

Let us verify the claim. Let $\mathcal{F}$ be the Fourier-Mukai transform of $L^\vee$. Then $\mathcal{F}$ is concentrated in cohomological degrees 0 and 1, and $H^0(\mathcal{F})$ is of pure dimension 1. Since $L^\vee$ is semistable,

$$\operatorname{Hom}(L^\vee, O_Y(-y)^\vee) = 0 \quad \text{for all but finitely many } y \in Y,$$
$$\operatorname{Hom}(O_Y(-y)^\vee, L^\vee) = 0 \quad \text{for all but finitely many } y \in Y.$$

Indeed, the Homs can be non-zero, only if $O_Y(-y)^\vee$ occurs among the Jordan–Holder factors of $L^\vee$.

Applying the Fourier-Mukai transform, we see that

$$\begin{aligned}\operatorname{Hom}(\mathcal{F}, O_y[-1]) = 0 &\quad \text{for all but finitely many } y \in Y,\\ \operatorname{Hom}(O_y[-1], \mathcal{F}) = 0 &\quad \text{for all but finitely many } y \in Y.\end{aligned}$$

The first statement implies that $H^1(\mathcal{F})$ is of finite length. Applying the Serre duality to the second statement, we see that

$$\operatorname{Hom}(\mathcal{F}, O_y) = 0 \quad \text{for all but finitely many } y \in Y,$$

hence $H^0(\mathcal{F}) = 0$.

In a similar way, Proposition 5.16 below can be proved using the Fourier-Mukai transform on $S \times Y$.

**Corollary 5.15.** $(a)$ *Let $\ell \in \overline{\mathcal{P}ic}_s^{\pm 1}(Y)$, then $\ell$ is a line bundle on the complement to a point.*
$(b)$ *Let $\ell \in \overline{\mathcal{P}ic}_s^{-1}(Y)$, then*

$$\dim H^i(\ell) = \begin{cases} 0 & \text{if } i = 0,\\ 1 & \text{if } i = 1.\end{cases}$$

$(c)$ *Let $\ell \in \overline{\mathcal{P}ic}_s^{1}(Y)$, then*

$$\dim H^i(\ell) = \begin{cases} 1 & \text{if } i = 0,\\ 0 & \text{if } i = 1.\end{cases}$$

*Proof.* The first part is clear. We have $H^0(O_Y(-y)) = 0$ thus $\dim H^1(O_Y(-y)) = -\chi(O_Y(-y)) = 1$. The last part follows from Serre duality. □

We are now ready to prove Proposition 5.7. Note that $\overline{\mathcal{P}ic}^{-1}(Y)$ is clearly locally Noetherian (it is of locally finite type over the ground field), so it suffices to prove the following

**Proposition 5.16.** *Let $S$ be a locally Noetherian scheme or stack, $\ell$ be an $S$-family of degree $-1$ stable generalized line bundles on $Y$ (that is, a coherent sheaf on $S \times Y$, flat over $S$, whose fibers are degree $-1$ stable generalized line bundles).*

*Then there exist a map $\psi : S \to Y$ and a line bundle $L$ on $S$ such that there exists an isomorphism*

$$\kappa : \ell \widetilde{\to} p_1^* L \otimes O_{S\times Y}(-\Gamma_\psi).$$

*Moreover, $\psi$ is unique, $L$ is unique up to isomorphism, and $\kappa$ is unique up to scaling by an element of $H^0(S, O_S^\times)$.*

*Proof.* By Lemma 5.10 and Corollary 5.11, the dual sheaf $\ell^\vee = \mathcal{H}om(\ell, O_{S\times Y})$ is an $S$-family of stable generalized line bundles on $Y$ of degree 1. Consider $Rp_{1,*}\ell^\vee$. By base change, we see that for every point $\iota_s : \{s\} \hookrightarrow S$,

$$\dim H^i(L\iota_s^* Rp_{1,*}\ell^\vee) = \dim H^i(\{s\} \times Y, \ell^\vee|_{\{s\}\times Y}) = \begin{cases} 1 & \text{if } i = 0,\\ 0 & \text{if } i \neq 0,\end{cases}$$

where the last equality follows from Corollary 5.15. Therefore, $p_{1,*}\ell^\vee = Rp_{1,*}\ell^\vee$ is a flat coherent sheaf with one-dimensional fibers, that is, a line bundle (the proof of flatness is as in Lemma 7.8). Let $L := (p_{1,*}\ell^\vee)^{-1}$ be its dual.

Consider the adjunction map

$$f : (p_1^* L)^{-1} \to \ell^\vee.$$

Denote by $U \subset S \times Y$ the open set where $f$ is surjective. Proposition 5.13 implies that $(\{s\} \times Y) \cap U \neq \emptyset$ for every $s \in S$. It is now easy to see that $f|_U$ is bijective, because $\ell^\vee$ is flat over $S$ and the restriction of $f$ to the fiber $\{s\} \times Y$ is injective for every $s \in S$. In particular $\ell^\vee|_U$ is a line bundle.

Consider the homomorphism $(p_1^* L)^{-1} \otimes \ell \to O_{S\times Y}$ induced by $f$. It is injective, because $\ell$ has no sections supported by $S \times Y - U$. Therefore, $(p_1^* L)^{-1} \otimes \ell$ is identified with the ideal sheaf of a closed subscheme $S' \subset S \times Y - U$.

We need to show that $p_1$ induces an isomorphism $S' \to S$. Recall that a projective quasi-finite morphism is finite (see [Har, Exercise III.11.2]). Thus $p_1|_{S'}$ is finite, so it suffices to prove that the natural map $O_S \to p_{1,*}O_{S'}$ is an isomorphism. Using base change and Proposition 5.13, we see that for every point $\iota_s : \{s\} \hookrightarrow S$, the map induces an isomorphism

$$\iota_s^* O_S \to L\iota_s^* p_{1,*} O_{S'}.$$

In other words, $p_{1,*}O_{S'}$ is flat and the map $O_S \to p_{1,*}O_{S'}$ is an isomorphism on fibers. Therefore, it is an isomorphism. Thus $S'$ is the graph of a map $\psi : S \to Y$.

The uniqueness statements are easy and left to the reader. □

5.3. **Fourier-Mukai transform.** It remains to prove Theorem 8. Actually, the argument of I. Burban and B. Kreussler from [BK] (for the case of an irreducible Weierstrass cubic, nodal or cuspidal) extends without difficulty to our settings, as mentioned in the introduction to [BK]. We sketch the argument and refer the reader to [BK] for details.

The key observation is that the structure sheaf $O_Y \in D^b(Y)$ is a *spherical object* in the sense of P. Seidel and R. Thomas [ST]. A spherical object $\mathcal{E} \in D^b(Y)$ defines an equivalence $T_\mathcal{E} : D^b(Y) \to D^b(Y)$ called the *twist functor*; roughly speaking, it sends $\mathcal{F} \in D^b(Y)$ to the cone of the evaluation morphism

$$R\operatorname{Hom}(\mathcal{E}, \mathcal{F}) \otimes^L \mathcal{E} \to \mathcal{F}.$$

(One has to replace $\mathcal{E}$ by its resolution to ensure that the cone is functorial.) We can then see that for the spherical object $\mathcal{E} = O_Y$, the twist functor $T_\mathcal{E}$ is the Fourier-Mukai transform of Theorem 8, cf. [BK, Proposition 2.10].

Let us translate this argument into the language of the Fourier-Mukai kernels, since we shall use Proposition 5.17 later. Let $p_{12}$, $p_{13}$, and $p_{23}$ be the usual projections $Y \times Y \times Y \to Y \times Y$. Set $\mathcal{F}_Y := Rp_{13,*}(p_{12}^* \mathcal{P}^\vee \otimes^L p_{23}^* \mathcal{P})$. Note that by Lemma 5.10, both $\mathcal{P}$ and $\mathcal{P}^\vee$ are flat with respect to both projections $Y \times Y \to Y$. In particular,

$$p_{12}^* \mathcal{P}^\vee \otimes^L p_{23}^* \mathcal{P} = p_{12}^* \mathcal{P}^\vee \otimes p_{23}^* \mathcal{P}$$

is a sheaf, so that $\mathcal{F}_Y$ is concentrated in cohomological dimensions 0 and 1.

**Proposition 5.17.**

$$\mathcal{F}_Y = O_\Delta[-1] \otimes_\mathbb{C} H^1(Y, O_Y) \simeq O_\Delta[-1].$$

*Proof.* Note first that

$$R\,\mathcal{H}om(O_\Delta, O_{Y\times Y}) = R\iota_\Delta^! O_{Y\times Y} = O_\Delta[-1] \otimes_\mathbb{C} H^1(Y, O_Y).$$

Here $\iota_\Delta : Y \to Y \times Y$ is the diagonal embedding. The second equality holds because $O_{Y\times Y}$ is the dualizing sheaf up to the second power of $H^1(Y, O_Y)$ (see Remark 5.2)

and the !-pullback of the dualizing sheaf is the dualizing sheaf up to a cohomological shift.

Applying the duality functor $R\mathcal{H}om(\cdot\,, O_{Y\times Y})$ to the exact sequence

$$0 \to \mathcal{P} \to O_{Y\times Y} \to O_\Delta \to 0, \tag{5.3}$$

we obtain an exact sequence

$$0 \to O_{Y\times Y} \to \mathcal{P}^\vee \to O_\Delta \otimes_{\mathbb{C}} H^1(Y, O_Y) \to 0. \tag{5.4}$$

It induces a map $O_Y = p_{1,*}O_{Y\times Y} \to p_{1,*}\mathcal{P}^\vee$; it is easy to see that the map provides an isomorphism $O_Y \widetilde{\to} Rp_{1,*}\mathcal{P}^\vee$ (see Corollary 5.15).

Finally, (5.3) also yields an exact sequence

$$0 \to p_{12}^*\mathcal{P}^\vee \otimes p_{23}^*\mathcal{P} \to p_{12}^*\mathcal{P}^\vee \to p_{12}^*\mathcal{P}^\vee \otimes p_{23}^*O_\Delta \to 0.$$

Applying $Rp_{13,*}$, we obtain a distinguished triangle

$$\mathcal{F}_Y \to (Rp_{1,*}\mathcal{P}^\vee) \boxtimes O_Y \to \mathcal{P}^\vee \to \mathcal{F}_Y[1].$$

It remains to notice that the map $(Rp_{1,*}\mathcal{P}^\vee) \boxtimes O_Y \to \mathcal{P}^\vee$ becomes the natural embedding $O_{Y\times Y} \to \mathcal{P}^\vee$ from (5.4) after the identification $Rp_{1,*}\mathcal{P}^\vee = O_Y$. □

It is easy to see that Proposition 5.17 implies Theorem 8: indeed, the base change shows, that the functors

$$\mathcal{D}(Y) \to \mathcal{D}(Y) : \mathcal{F} \mapsto Rp_{1,*}(\mathcal{P} \otimes^L p_2^*\mathcal{F})$$

and

$$\mathcal{D}(Y) \to \mathcal{D}(Y) : \mathcal{F} \mapsto Rp_{2,*}(\mathcal{P}^\vee \otimes^L p_1^*\mathcal{F}) \otimes_{\mathbb{C}} (H^1(Y, O_Y))^{-1}[1]$$

are mutual inverses.

## 6. Geometric description of $\mathcal{M}_H$

Recall that our goal is to calculate cohomology of certain natural vector bundle on $\mathcal{M}$ (or more precisely, a direct image, see Theorem 4). In this section we calculate the direct image of the extension of this sheaf to $\mathcal{M}_H$ (see §4.2 for the definition of $\mathcal{M}_H$). The main result is Proposition 6.10. The calculation is based on explicit identification of $\mathcal{M}_H$, see Corollary 6.7, and applying the Fourier–Mukai transform.

We claim that $\mathcal{M}_H$ is the moduli stack of collections $(L, \nabla; E)$, where $L$ is a rank 2 degree $d$ vector bundle on $\mathbb{P}^1$, $E$ is a one-dimensional vector space, $\nabla : L \to L \otimes \Omega_{\mathbb{P}^1}(\mathfrak{D}) \otimes_{\mathbb{C}} E$ is an $O_{\mathbb{P}^1}$-linear morphism, satisfying the following conditions:
(a) $\nabla$ is not nilpotent, that is $\nabla^2 \neq 0$.
(b) $\operatorname{tr}\nabla = 0$.
(c) $\det\nabla$ is a section of $E^{\otimes 2} \otimes_{\mathbb{C}} \Omega_{\mathbb{P}^1}^{\otimes 2}(\mathfrak{D})$.
(d) $(L, \nabla; E)$ is semistable.
Note that $\operatorname{tr}\nabla$ is a section of $E \otimes_{\mathbb{C}} \Omega_{\mathbb{P}^1}(\mathfrak{D})$. It follows from (4.2) that $\operatorname{tr}\nabla$ is in fact a section of $E \otimes_{\mathbb{C}} \Omega_{\mathbb{P}^1}$, which implies condition (b). Condition (c) is a condition on the polar part of $\nabla$: a priori $\det\nabla$ is in $E^{\otimes 2} \otimes_{\mathbb{C}} \Omega_{\mathbb{P}^1}^{\otimes 2}(2\mathfrak{D})$. This condition also follows from (4.2).

Note that $\nabla^2 = -\det\nabla \otimes \mathrm{id}_L$. Recall that $\mathcal{E}$ is the line bundle on $\overline{\mathcal{M}}$ whose fiber at $(L, \nabla; \varepsilon \in E)$ is $E$. For simplicity we write $\mathcal{E}$ for $\mathcal{E}|_{\mathcal{M}_H}$. The following statement follows from (a) and (c) above

**Lemma 6.1.**

$$\mathcal{E}^{\otimes 2}|_{\mathcal{M}_H} \simeq O_{\mathcal{M}_H}.$$

Let us fix a global section $\mu$ of $\Omega_{\mathbb{P}^1}^{\otimes 2}(\mathfrak{D}) \simeq O_{\mathbb{P}^1}$, $\mu \neq 0$. One can choose an isomorphism $E \simeq \mathbb{C}$ such that $\det \nabla = \mu$ (there are two choices for such an isomorphism). Denote by $\mathcal{Y}$ the moduli stack of pairs $(L, \nabla)$, where $L$ is a rank 2 degree $d$ vector bundle on $\mathbb{P}^1$, $\nabla \in H^0(\mathbb{P}^1, \mathcal{E}nd(L) \otimes \Omega_{\mathbb{P}^1}(\mathfrak{D}))$, $\operatorname{tr} \nabla = 0$, $\det \nabla = \mu$, and the pair $(L, \nabla)$ is semistable. We have proved the following

**Proposition 6.2.** *The correspondence $(L, \nabla) \mapsto (L, \nabla; 0 \in \mathbb{C})$ yields a double cover $\mathcal{Y} \to \mathcal{M}_H$. Besides, $\mathcal{M}_H$ is identified with the quotient stack $\mu_2 \backslash \mathcal{Y}$, where $\pm 1 \in \mu_2$ acts on $\mathcal{Y}$ by $(L, \nabla) \mapsto (L, \pm\nabla)$.*

It follows directly from the definition of $\mathcal{Y}$ that the pullback of $\mathcal{E}$ to $\mathcal{Y}$ is $O_{\mathcal{Y}}$.

Set $\mathcal{A} := O_{\mathbb{P}^1} \oplus \Omega_{\mathbb{P}^1}(\mathfrak{D})^{-1}$. Then $\mathcal{A}$ is a sheaf of $O_{\mathbb{P}^1}$-algebras with respect to the multiplication

$$(f_1, \tau_1) \times (f_2, \tau_2) := (f_1 f_2 - \mu \otimes \tau_1 \otimes \tau_2, f_1 \tau_2 + f_2 \tau_1).$$

Set $\pi : Y := \mathcal{S}pec(\mathcal{A}) \to \mathbb{P}^1$. Denote by $y_i \in Y$ the preimage of $x_i \in \mathbb{P}^1$, and by $\sigma : Y \to Y$ the involution induced by $\sigma^* : \mathcal{A} \to \mathcal{A} : (f, \tau) \mapsto (f, -\tau)$.

**Proposition 6.3.** *$Y$ is a generalized elliptic curve.*

*Proof.* Since $\pi$ is a finite morphism, $Y$ has dimension 1. The dualizing complex of $Y$ is given by $\mathcal{H}om(\mathcal{A}, \Omega_{\mathbb{P}^1})$. Thus we need to show that this sheaf is isomorphic to $\mathcal{A}$ as an $\mathcal{A}$-module. It is clear on the level of $O_{\mathbb{P}^1}$-modules, since $\mathcal{A} \simeq O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-2)$. Let $\gamma \in \mathcal{H}om(\mathcal{A}, \Omega_{\mathbb{P}^1})$ be the composition of the projection $\mathcal{A} \to (\Omega_{\mathbb{P}^1}(\mathfrak{D}))^{-1}$ and an isomorphism. One checks easily that the map of $\mathcal{A}$-modules $\mathcal{A} \to \mathcal{H}om(\mathcal{A}, \Omega_{\mathbb{P}^1})$ given by $1 \mapsto \gamma$ is injective. Now, an injective map of a vector bundle to an isomorphic one is necessarily an isomorphism.

Also, $H^0(Y, O_Y) = H^0(\mathbb{P}^1, \mathcal{A}) = \mathbb{C}$, thus $Y$ is generalized elliptic.

□

*Remark* 6.4. Actually $Y$ is always reduced. Precisely, $Y$ is a smooth elliptic curve if $\mathfrak{D}$ has no multiple points; $Y$ is a nodal cubic if $\mathfrak{D}$ has a single multiple point of multiplicity 2; $Y$ is a cuspidal cubic if $\mathfrak{D} = 3(x_1) + (x_2)$; $Y$ has two components, isomorphic to $\mathbb{P}^1$, which intersect transversally at two points if $\mathfrak{D} = 2(x_1) + 2(x_2)$; and $Y$ has two components, isomorphic to $\mathbb{P}^1$, which are tangent to each other if $\mathfrak{D} = 4(x_1)$.

**Proposition 6.5.** *$\mathcal{Y}$ is naturally isomorphic to $\overline{\mathcal{P}ic}_s^{\,d+2} Y$, that is the moduli stack of stable generalized line bundles of degree $d+2$ on $Y$.*

*Proof.* Let $(L, \nabla)$ be a point of $\mathcal{Y}$. Then $L$ is an $\mathcal{A}$-module with respect to the multiplication $(f, \tau)s := fs + \tau \otimes \nabla s$, let us denote the corresponding sheaf on $Y$ by $\ell$. It is a standard fact about the Hitchin system that $\ell$ is a generalized line bundle on $Y$. The inverse construction is given by $\ell \mapsto L := \pi_* \ell$.

Let the weight function $w$ from §5 be given by the degree of the projection $\pi : Y \to \mathbb{P}^1$. Then $\operatorname{rk} \pi_* \ell_0 = \operatorname{rk} \ell_0$ for any coherent sheaf $\ell_0$ on $Y$.

We would like to show that $\ell$ is stable if and only if $(L, \nabla)$ is stable. Note that $\nabla$-invariant subsheaves of $L$ are in bijection with subsheaves $\ell_0 \subset \ell$ via $\ell_0 \mapsto \pi_* \ell_0$. Further,

$$\deg \ell_0 = \chi(\ell_0) = \chi(\pi_* \ell_0) = \deg \pi_* \ell_0 + \operatorname{rk} \pi_* \ell_0. \tag{6.1}$$

It follows that the stability condition is the same.

It also follows from (6.1) that the generalized line bundles on $Y$ corresponding to rank 2 degree $d$ bundles on $\mathbb{P}^1$ have degree $d+2$. $\square$

*Remark* 6.6. If $\mathfrak{D}$ is not even, then $Y$ is integral, and for every $\ell_0 \subset \ell$, $\ell_0 \neq 0$ we have $\operatorname{rk} \ell_0 = \operatorname{rk} \ell$ thus the semistability condition is trivial.

Fix a degree $(d+3)/2$ line bundle $\vartheta$ on $\mathbb{P}^1$ (recall that $d$ is odd). A Higgs bundle $(L, \nabla)$ is semistable if and only if $(L \otimes \vartheta, \nabla)$ is. Therefore, Proposition 5.7 implies the following

**Corollary 6.7.** *Consider the map $Y \to \mathcal{Y}$ that sends $y \in Y$ to the vector bundle $\vartheta \otimes \pi_* O_Y(-y)$ equipped with the natural Higgs field. The map induces an isomorphism*

$$Y \times B(\mathbf{G_m}) \xrightarrow{\sim} \mathcal{Y}.$$

*Thus $\mathcal{M}_H$ is the quotient of the generalized elliptic curve $Y$ by the action of $\mu_2 \times \mathbf{G_m}$, where $\mu_2$ acts by $\sigma$, $\mathbf{G_m}$ acts trivially.* $\square$

Let us use the isomorphism of Corollary 6.7 to describe the universal Higgs bundle on $\mathbb{P}^1 \times \mathcal{M}_H$. Denote this universal Higgs bundle by $\xi$ and its pullback to $\mathbb{P}^1 \times Y$ by $\tilde{\xi} := (\mathrm{id}_{\mathbb{P}^1} \times \bar{\pi})^* \xi$ (here $\bar{\pi}$ is the natural composition $Y \to \mathcal{Y} \to \mathcal{M}_H$). Recall also that $\mathcal{P} := O_{Y \times Y}(-\Delta)$ is the ideal sheaf of the diagonal.

**Corollary 6.8.** *We have $\tilde{\xi} = (\pi \times \mathrm{id}_Y)_* \mathcal{P} \otimes p_1^* \vartheta$. The action of $\mathbf{G_m}$ on $\tilde{\xi}$ is via the identity character $a \mapsto a$ and the action of $\mu_2$ comes from its action on $\mathcal{P}$ (on $Y \times Y$, $-1 \in \mu_2$ acts as $\sigma^* \times \sigma^*$).*

*For the dual bundle,*

$$\tilde{\xi}^\vee = (\pi \times \mathrm{id}_Y)_* \mathcal{P}^\vee \otimes p_1^*(\vartheta^\vee \otimes \mathcal{T}_{\mathbb{P}^1}) \otimes_{\mathbb{C}} H^1(Y, O_Y)^\vee.$$

*On this bundle, $\mathbf{G_m}$ acts via the character $a \mapsto a^{-1}$ and the action of $\mu_2$ comes from its action on $\mathcal{P}$ and its action on $H^1(Y, O_Y)$ (by $-1$).*

*Proof.* The description of $\tilde{\xi}$ follows from Proposition 6.5 and Corollary 6.7, and then the description of $\tilde{\xi}^\vee$ follows from Serre duality. $\square$

*Remark* 6.9. It is easy to describe the Fourier-Mukai transform of $\tilde{\xi}$: this is the structure sheaf of the graph of $\pi$ twisted by $\vartheta$.

Consider now the sheaf $\mathcal{F}_H := p_{13}^* \xi \otimes p_{23}^* \xi^\vee$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{M}_H$. The main result of this section is the following

**Proposition 6.10.**

$$Rp_{12,*}(\mathcal{F}_H \otimes p_3^* \mathcal{E}^{\otimes k}) \simeq \begin{cases} \iota_{\Delta,*}(\mathcal{T}_{\mathbb{P}^1})[-1] & \text{if } k \text{ is even,} \\ \iota_{\Delta,*}(\mathcal{T}_{\mathbb{P}^1}^{\otimes 2}(-\mathfrak{D}))[-1] & \text{if } k \text{ is odd,} \end{cases}$$

*where $\iota_\Delta : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is the diagonal embedding.*

*Proof.* The pullback $(\mathrm{id}_{\mathbb{P}^1 \times \mathbb{P}^1} \times \bar{\pi})^*(p_{13}^* \xi \otimes p_{23}^* \xi^\vee)$ to $\mathbb{P}^1 \times \mathbb{P}^1 \times Y$ equals $p_{13}^* \tilde{\xi} \otimes p_{23}^* \tilde{\xi}^\vee$. By Corollary 6.8 and Proposition 5.17, we have the following identity on $\mathbb{P}^1 \times \mathbb{P}^1$

$$\begin{aligned} &Rp_{12,*}(p_{13}^* \tilde{\xi} \otimes p_{23}^* \tilde{\xi}^\vee) = \\ &\quad (\pi \times \pi)_* Rp_{12,*}(p_{13}^* \mathcal{P} \otimes p_{23}^* \mathcal{P}^\vee) \otimes p_1^* \vartheta \otimes p_2^*(\vartheta^\vee \otimes \mathcal{T}_{\mathbb{P}^1}) \otimes_{\mathbb{C}} H^1(Y, O_Y)^\vee = \\ &\qquad \iota_{\Delta,*}(\pi_* O_Y \otimes \mathcal{T}_{\mathbb{P}^1})[-1]. \end{aligned}$$

The action of $\mathbf{G_m}$ on the right-hand side is trivial. The action of $\mu_2$ on $O_Y$ is the standard action coming from $\sigma : Y \to Y$ (in other words, $-1 \in \mu_2$ acts by $\sigma^*$).

Note that $\bar{\pi}^*\mathcal{E} = O_Y$, but $-1 \in \mu_2$ acts on $\bar{\pi}^*\mathcal{E}$ as $-\sigma^*$ (and $\mathbf{G_m}$ acts trivially). Since $\mathcal{M}_H = Y/(\mathbf{G_m} \times \mu_2)$,

$$Rp_{12,*}(\mathcal{F}_H \otimes p_3^*\mathcal{E}^{\otimes k}) = \Big(Rp_{12,*}((\mathrm{id}_{\mathbb{P}^1\times\mathbb{P}^1} \times \bar{\pi})^*(\mathcal{F}_H \otimes p_3^*\mathcal{E}^{\otimes k}))\Big)^{\mathbf{G_m}\times\mu_2}$$
$$\simeq \begin{cases} (\iota_{\Delta,*}(\pi_* O_Y \otimes \mathcal{T}_{\mathbb{P}^1}))^{(1)}[-1] & \text{if } k \text{ is even,} \\ (\iota_{\Delta,*}(\pi_* O_Y \otimes \mathcal{T}_{\mathbb{P}^1}))^{(-1)}[-1] & \text{if } k \text{ is odd.} \end{cases}$$

Here for a sheaf $\mathcal{V}$ with an action of $\mu_2$, we denote by $\mathcal{V}^{(1)}$ (resp. $\mathcal{V}^{(-1)}$) its eigensheaf on which $-1 \in \mu_2$ acts as 1 (resp. $-1$). Finally,

$$\pi_* O_Y = \mathcal{A} = O_{\mathbb{P}^1} \oplus \Omega_{\mathbb{P}^1}(\mathfrak{D})^{-1},$$

and $-1 \in \mu_2$ acts on $O_{\mathbb{P}^1}$ as 1 and on $\Omega_{\mathbb{P}^1}(\mathfrak{D})^{-1}$ as $-1$. □

## 7. First orthogonality relation

In this section, we prove Theorem 4.

7.1. Recall that $\mathcal{F}_P = p_{13}^*\xi_\alpha \otimes p_{23}^*\xi_\alpha^\vee$ is a quasi-coherent sheaf on $P \times P \times \mathcal{M}$ equipped with an action of $\mathcal{D}_{P,\alpha}$ along the first copy of $P$ and an action of $\mathcal{D}_{P,-\alpha}$ along the second. Accordingly, the direct image $Rp_{12,*}\mathcal{F}_P$ is an object of the derived category of $p_1^\bullet\mathcal{D}_{P,\alpha} \circledast p_2^\bullet\mathcal{D}_{P,-\alpha}$-modules on $P \times P$.

Let $\iota_\Delta : P \to P\times P$ be the diagonal embedding. Recall that $\delta_\Delta$ is a $\mathcal{D}_{P,\alpha}\boxtimes\mathcal{D}_{P,-\alpha}$-module given by $\iota_{\Delta,*}O_P$.

**Lemma 7.1.** *In the category of $\mathcal{D}_{P,\alpha} \boxtimes \mathcal{D}_{P,-\alpha}$-modules we have*

$$\delta_\Delta = \iota_{\Delta,*}O_P = \iota_{\Delta,!}O_P.$$

*Proof.* Let $\Delta$ be the diagonal in $P \times P$ and $\overline{\Delta}$ be its closure. We can decompose $\iota_\Delta$ as

$$\Delta \xrightarrow{\iota_1} \overline{\Delta} \xrightarrow{\iota_2} P \times P.$$

Since $\iota_2$ is a closed embedding, we have $\iota_{2,*} = \iota_{2,!}$. Thus it is enough to show that

$$\iota_{1,*}O_P = \iota_{1,!}O_P.$$

Note that $\iota_1$ is an open embedding, $\overline{\Delta} - \Delta$ consists of 8 points, and twists at these points are given by $\pm(\alpha_i^+ - \alpha_i^-)$. Now the statement follows from conditions (a) and (d) of §2.1. Note that $\iota_{\Delta,*}$ and $\iota_{\Delta,!}$ are exact functors, since $\iota_\Delta$ is an affine embedding. □

Further, the restriction $(\iota_\Delta \times \mathrm{id}_\mathcal{M})^*\mathcal{F}_P$ is a quasi-coherent sheaf on $P \times \mathcal{M}$ equipped with a structure of a $\mathcal{D}_P$-module. Recall that $\wp : P \to \mathbb{P}^1$ is the natural projection. It is easy to see that we have a natural inclusion $\wp^*\xi \subset \xi_\alpha$ (see Remark 2.2). Thus, the identity automorphism of $\xi$ gives a horizontal section

$$1 \in H^0(P \times \mathcal{M}, (\iota_\Delta \times \mathrm{id}_\mathcal{M})^*(p_{13}^*\wp^*\xi \otimes p_{23}^*\wp^*\xi^\vee)) \subset H^0(P \times \mathcal{M}, (\iota_\Delta \times \mathrm{id}_\mathcal{M})^*\mathcal{F}_P).$$

We thus obtain a horizontal section of $p_{1,*}(\iota_\Delta \times \mathrm{id}_\mathcal{M})^*\mathcal{F}_P$, which can be viewed as a morphism

$$O_P \to Rp_{1,*}(\iota_\Delta \times \mathrm{id}_\mathcal{M})^*\mathcal{F}_P = \iota_\Delta^* Rp_{12,*}\mathcal{F}_P$$

in the derived category of $\mathcal{D}_P$-modules (we use base change). Finally, adjunction provides a morphism

$$\varphi : \delta_\Delta[-1] = \iota_{\Delta,!}O_P[-1] \to Rp_{12,*}\mathcal{F}_P$$

(we are using Lemma 7.1). Note that the appearance of the shift $[-1]$ is due to the fact that our inverse images are $O$-module inverse images; from the point of view of $\mathcal{D}$-modules they should contain shifts.

Theorem 4 claims that $\varphi$ is an isomorphism. We derive Theorem 4 from two statements that are proved later in this section.

**Proposition 7.2.** *The direct image $R(\wp \times \wp)_* Rp_{12,*}\mathcal{F}_P$ vanishes outside the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$.*

**Proposition 7.3.** *Consider the morphism*

$$H^1(\varphi) : \delta_\Delta \to R^1 p_{12,*}\mathcal{F}_P$$

*induced by $\varphi$. Then its cokernel is such that $(\wp \times \wp)_* \operatorname{Coker}(H^1(\varphi))$ is coherent.*

*Remark* 7.4. Note that in Proposition 7.3, we consider naive (not derived) direct image $(\wp \times \wp)_*$. Actually, higher derived images $R^i(\wp \times \wp)_*\mathcal{G}$ $(i > 0)$ vanish for any $p_1^\bullet\mathcal{D}_{P,\alpha} \circledast p_2^\bullet\mathcal{D}_{P,-\alpha}$-module $\mathcal{G}$ (see Remark 7.10(iii)).

*Proof of Theorem 4.* By Proposition 4.1, we see that $R^i p_{12,*}\mathcal{F}_P = 0$ for all $i \neq 0, 1$. Also, $R^0 p_{12,*}\mathcal{F}_P$ vanishes at the generic point by Proposition 7.2, which implies $R^0 p_{12,*}\mathcal{F}_P = 0$. Thus $Rp_{12,*}\mathcal{F}_P$ is concentrated in cohomological dimension one. It remains to show that $H^1(\varphi)$ is an isomorphism.

By construction, $\varphi \neq 0$. Since $\delta_\Delta$ is irreducible as a $p_1^\bullet\mathcal{D}_{P,\alpha} \circledast p_2^\bullet\mathcal{D}_{P,-\alpha}$-module, $H^1(\varphi)$ is injective. Its cokernel $\mathcal{F}' := \operatorname{Coker}(H^1(\varphi))$ is a $p_1^\bullet\mathcal{D}_{P,\alpha} \circledast p_2^\bullet\mathcal{D}_{P,-\alpha}$-module such that $(\wp \times \wp)_*\mathcal{F}'$ is a coherent sheaf (by Proposition 7.3) that vanishes generically (by Proposition 7.2). It is now easy to see that $\mathcal{F}' = 0$.

Indeed, consider a stratification of $P \times P$ by sets of the form $\{(x_i^\pm, x_j^\pm)\}$, $\{x_i^\pm\} \times (\mathbb{P}^1 - \mathfrak{D})$, $(\mathbb{P}^1 - \mathfrak{D}) \times \{x_i^\pm\}$, and $(\mathbb{P}^1 - \mathfrak{D}) \times (\mathbb{P}^1 - \mathfrak{D})$. We can now show that $\mathcal{F}'$ vanishes on all strata by descending induction on the dimension of strata. □

## 7.2. **Proof of Proposition 7.2.**

**Lemma 7.5.**

$$R(\wp \times \mathrm{id}_{\mathcal{M}})_*\xi_\alpha = \xi.$$

*Proof.* The sheaves are obviously identified on $(\mathbb{P}^1 - \mathfrak{D}) \times \mathcal{M}$, so it remains to verify that this identification extends to $\mathbb{P}^1 \times \mathcal{M}$. It suffices to check this on $D \times \mathcal{M}$, where $D$ is the formal neighborhood of $x_i$. The restriction $\xi|_{D\times\mathcal{M}}$ decomposes into a direct sum $\xi^+ \oplus \xi^-$ of one-dimensional bundles that are invariant under the connection (that acts in the direction of $D$). This can be viewed as a version of diagonalization (2.1).

The preimage $\wp^{-1}D$ is a union of two copies of $D$ glued away from the center, and the restrictions of $\xi_\alpha$ to $\wp^{-1}D$ is of the form $(j_+ \times \mathrm{id}_{\mathcal{M}})_*\xi_+ \oplus (j_- \times \mathrm{id}_{\mathcal{M}})_*\xi_-$, where $j_\pm : D \to \wp^{-1}D$ are the embeddings of the two copies (see Remark 2.2). Since $\wp \circ j_\pm = \mathrm{id}_D$, the claim follows. □

Consider now the sheaf $\mathcal{F}_{\mathbb{P}^1} := p_{13}^*\xi \otimes p_{23}^*\xi^\vee$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{M}$.

**Corollary 7.6.**

$$R(\wp \times \wp \times \mathrm{id}_{\mathcal{M}})_*\mathcal{F}_P = \mathcal{F}_{\mathbb{P}^1}.$$

*Proof.* This follows from Lemma 7.5 and a similar statement about $\xi_\alpha^\vee$ upon writing $\mathcal{F}_P = \Delta_{24}^*(\xi_\alpha \boxtimes \xi_\alpha^\vee)$, where $\Delta_{24} : P \times P \times \mathcal{M} \to P \times \mathcal{M} \times P \times \mathcal{M}$ is a partial diagonal. □

By Corollary 7.6,

$$R(\wp \times \wp)_* Rp_{12,*}\mathcal{F}_P = Rp_{12,*}\mathcal{F}_{\mathbb{P}^1}. \tag{7.1}$$

The advantage of working with $\xi$ rather than $\xi_\alpha$ is that $\xi$ is naturally defined as a vector bundle (the universal bundle) on $\mathbb{P}^1 \times \overline{\mathcal{M}}$. Accordingly, $\mathcal{F}_{\mathbb{P}^1}$ extends to a vector bundle $\overline{\mathcal{F}} := p_{13}^*\xi \otimes p_{23}^*\xi^\vee$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \overline{\mathcal{M}}$. Set

$$\mathcal{F}_k := \overline{\mathcal{F}}(k(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{M}_H)), \qquad k \in \mathbb{Z}.$$

Let $\jmath : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{M} \to \mathbb{P}^1 \times \mathbb{P}^1 \times \overline{\mathcal{M}}$ be the natural embedding. In view of Proposition 4.7, we have a filtration

$$\mathcal{F}_0 = \overline{\mathcal{F}} \subset \cdots \subset \mathcal{F}_k \subset \cdots \subset \mathcal{F}_\infty := \jmath_*\mathcal{F}_{\mathbb{P}^1}. \tag{7.2}$$

We shall use notation $\Delta$ for diagonals in $P \times P$ and $\mathbb{P}^1 \times \mathbb{P}^1$ through the end of this section.

**Lemma 7.7.** *For any $k$ and $i$ there is an isomorphism*

$$(R^i p_{12,*}\mathcal{F}_{\mathbb{P}^1})|_{\mathbb{P}^1\times\mathbb{P}^1-\Delta} \simeq (R^i p_{12}\mathcal{F}_k)|_{\mathbb{P}^1\times\mathbb{P}^1-\Delta}.$$

*Proof.* For every $k$ we have the short exact sequence

$$0 \to \mathcal{F}_{k-1} \to \mathcal{F}_k \to \iota_*(\mathcal{F}_k|_{\mathbb{P}^1\times\mathbb{P}^1\times\mathcal{M}_H}) \to 0,$$

where $\iota : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathcal{M}_H \to \mathbb{P}^1 \times \mathbb{P}^1 \times \overline{\mathcal{M}}$ is the closed embedding. Since

$$\mathcal{F}_k|_{\mathbb{P}^1\times\mathbb{P}^1\times\mathcal{M}_H} = \mathcal{F}_H \otimes p_3^*\mathcal{E}^{\otimes k},$$

Proposition 6.10 implies that $Rp_{12,*}(\mathcal{F}_k/\mathcal{F}_{k-1}) = 0$ away from the diagonal, so $Rp_{12,*}\mathcal{F}_k = Rp_{12,*}\mathcal{F}_{k-1}$ away from the diagonal. Now the claim follows from the identity $R^i p_{12,*}\mathcal{F}_{\mathbb{P}^1} = \varinjlim R^i p_{12,*}\mathcal{F}_k$. □

*Proof of Proposition 7.2.* Consider $(x,y) \in \mathbb{P}^1 \times \mathbb{P}^1 - \Delta$. We have

$$H^i(\overline{\mathcal{M}}, \overline{\mathcal{F}}|_{(x,y)}) = H^i(\mathcal{M}, \mathcal{F}_{\mathbb{P}^1}|_{(x,y)}) = 0 \tag{7.3}$$

for $i \geq 2$ by Proposition 4.1. Next, $H^0(\overline{\mathcal{M}}, \overline{\mathcal{F}}|_{(x,y)})$ is finite-dimensional because the good moduli space of $\overline{\mathcal{M}}$ is projective. It follows that $H^0(\overline{\mathcal{M}}, \mathcal{F}_k|_{(x,y)}) = 0$ for $k \ll 0$ because $\mathcal{M}$ is connected and $\mathcal{F}_{\mathbb{P}^1}$ is a vector bundle. Therefore Lemma 7.7 implies that

$$H^0(\overline{\mathcal{M}}, \overline{\mathcal{F}}|_{(x,y)}) = H^0(\mathcal{M}, \mathcal{F}_{\mathbb{P}^1}|_{(x,y)}) = 0. \tag{7.4}$$

It remains to show that $\mathcal{G} := (R^1 p_{12,*}\mathcal{F}_{\mathbb{P}^1})|_{\mathbb{P}^1\times\mathbb{P}^1-\Delta}$ vanishes. By Lemma 7.7, $\mathcal{G} = R^1 p_{12,*}\overline{\mathcal{F}}|_{\mathbb{P}^1\times\mathbb{P}^1-\Delta}$. Moreover, (7.3) and (7.4) imply that $\mathcal{G}$ is a vector bundle on $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta$. Its rank can be computed using the Euler characteristic; it equals $-\chi(\overline{\mathcal{M}}, \xi_x \otimes \xi_y^\vee)$ for any $x, y \in \mathbb{P}^1$. (Here $\xi_x$ is the restriction of $\xi$ to $\{x\} \times \overline{\mathcal{M}}$.)

Recall from §4.2 that the stack $\overline{\mathcal{M}}$ depends on the divisor $\mathfrak{D}$ and the formal type, which we encode by $\nu_1 \in \Omega_{\mathbb{P}^1}(\mathfrak{D})/\Omega_{\mathbb{P}^1}$ and $\nu_2 \in \Omega_{\mathbb{P}^1}^{\otimes 2}(2\mathfrak{D})/\Omega_{\mathbb{P}^1}^{\otimes 2}(\mathfrak{D})$. By Proposition 4.14, as the parameters vary, stacks $\overline{\mathcal{M}}$ form a flat family $\overline{\mathcal{M}}_{univ}$ over a Zariski open subspace in the space of collections $(\mathfrak{D}, \nu_1, \nu_2)$. The vector bundle $\xi_x \otimes \xi_y^\vee$ makes sense in this family. We shall use

**Lemma 7.8.** *Let $\mathcal{X} \to S$ be a flat family of stacks over a scheme $S$. Let $\mathcal{F}$ be a flat sheaf on $\mathcal{X}$. For $s \in S$ denote by $\mathcal{F}_s$ the fiber over $s$. Assume that there is a good moduli space $p : \mathcal{X} \to X$ such that the induced map $X \to S$ is projective.*

*Then $\chi(\mathcal{F}_s)$ is locally constant as a function of $s$.*

Let us apply this Lemma to $\xi_x \otimes \xi_y^\vee$. A slight generalization of Theorems 6 and 7 shows that $\overline{\mathcal{M}}_{univ}$ has a good moduli space $\overline{M}_{univ}$, which is projective over the space of collections $(\mathfrak{D}, \nu_1, \nu_2)$. Therefore, $\chi(\overline{\mathcal{M}}, \xi_x \otimes \xi_y^\vee)$ does not depend on $\mathfrak{D}$, $\nu_1$, and $\nu_2$. In particular, we may assume that $\mathfrak{D} = x_1 + x_2 + x_3 + x_4$ for distinct $x_i \in \mathbb{P}^1$ and that $\nu_1$, $\nu_2$ are generic. Using Lemma 7.7, we see that it is enough to prove that $H^i(\mathcal{M}, \xi_x \otimes \xi_y^\vee) = 0$ for $x \neq y$ and all $i$ in the case of simple $\mathfrak{D}$ and generic $\nu_1$ and $\nu_2$. This case is treated in [Ari2, Theorem 2], except for a slight difference that $SL(2)$-bundles are considered there. However, both moduli stacks have the same good moduli space, so the cohomology groups are the same (in fact, our moduli space is $M \times B(\mathbf{G_m})$, while the moduli space in [Ari2] is a $\mu_2$-gerbe over $M$. □

*Proof of Lemma 7.8.* Note that $p_*\mathcal{F}$ is flat on $X$. Indeed, if $\mathcal{G}$ is a sheaf on $X$, and $\mathcal{G}^\bullet$ is its resolution by locally free sheaves, we have

$$\mathcal{T}or^i(\mathcal{G}, p_*\mathcal{F}) = H^{-i}(\mathcal{G}^\bullet \otimes p_*\mathcal{F}) = p_* H^{-i}(p^*\mathcal{G}^\bullet \otimes \mathcal{F}) = 0.$$

We have used the projection formula and the fact that good moduli spaces are cohomologically affine.

By Proposition 4.7(i) of [Alp], the restriction of $p$ to $s \in S$ is a good moduli space $p_s : \mathcal{X}_s \to X_s$ so we have

$$\chi(\mathcal{F}_s) = \chi(p_{s,*}\mathcal{F}_s) = \chi((p_*\mathcal{F})_s).$$

(We are using a base change).

By Theorem 4.16(ix) of [Alp], the map $X \to S$ is flat, and we see that $\chi((p_*\mathcal{F})_s)$ is locally constant. □

7.3. **Proof of Proposition 7.3.** It is convenient to replace $\mathcal{D}_{P,\alpha}$-modules with modules over a certain sheaf of algebras on $\mathbb{P}^1$. Let us make the corresponding definitions. We identify $\mathcal{D}_{P,\alpha}$ with a subsheaf in the pushforward of $\mathcal{D}_{\mathbb{P}^1 - \mathfrak{D}}$ to $P$.

Let $z_i$ be a local coordinate at $x_i$. Let us lift polar parts $\alpha_i^\pm$ to actual 1-forms on formal neighborhoods of $x_i$; we shall denote these 1-forms by the same letters. Consider the open embedding $\jmath : \mathbb{P}^1 - \mathfrak{D} \hookrightarrow \mathbb{P}^1$.

Define the sheaf of algebras $\mathcal{B} = \mathcal{B}_\alpha \subset \jmath_*\mathcal{D}_{\mathbb{P}^1 - \mathfrak{D}}$ as follows:

- We have $\mathcal{B}|_{\mathbb{P}^1 - \mathfrak{D}} = \mathcal{D}_{\mathbb{P}^1 - \mathfrak{D}}$;
- Near $x_i \in \mathbb{P}^1$, $\mathcal{B}$ is generated by $O_{\mathbb{P}^1}$, $z_i^{n_i}\frac{\mathbf{d}}{\mathbf{d}z_i}$, and $z_i^{-n_i}(z_i^{n_i}\frac{\mathbf{d}-\alpha_i^+}{\mathbf{d}z_i})(z_i^{n_i}\frac{\mathbf{d}-\alpha_i^-}{\mathbf{d}z_i})$.

Clearly, $\mathcal{B}$ inherits from $\jmath_*\mathcal{D}_{\mathbb{P}^1 - \mathfrak{D}}$ the filtration by degree of differential operators. We denote by $\mathcal{B}^{\leq k} \subset \mathcal{B}$ the subsheaf of operators of degree at most $k$.

The properties of $\mathcal{B}$ are summarized in the following

**Proposition 7.9.** $(a)$ $\mathcal{B} = \wp_*\mathcal{D}_{P,\alpha}$.
$(b)$ *Moreover,* $R^1\wp_*\mathcal{D}_{P,\alpha} = 0$, *so that* $\mathcal{B} = R\wp_*\mathcal{D}_{P,\alpha}$.
$(c)$ $\mathcal{B}^{\leq k}/\mathcal{B}^{\leq k-1} = \mathcal{T}_{\mathbb{P}^1}^{\otimes k}(-\left\lceil \frac{k}{2} \right\rceil \mathfrak{D})$. *(Here* $\lceil\,\rceil$ *is the ceiling function.)*

*Remark* 7.10. (i) The isomorphisms (a) and (c) are naturally normalized by the condition that they become the obvious identifications on $\mathbb{P}^1 - \mathfrak{D}$.
(ii) Let us fix $\mu \in H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^{\otimes 2}(\mathfrak{D}))$, $\mu \neq 0$, as in §6. Then (c) can be rewritten as

$$\mathcal{B}^{\leq k}/\mathcal{B}^{\leq k-1} = \begin{cases} \mathcal{T}_{\mathbb{P}^1}(-\mathfrak{D}) & \text{if } k \text{ is odd,} \\ O_{\mathbb{P}^1} & \text{if } k \text{ is even.} \end{cases}$$

(iii) Actually, $\wp : P \to \mathbb{P}^1$ is affine with respect to $\mathcal{D}_{P,\alpha}$ in the sense that the functor $\wp_*$ is exact on $\mathcal{D}_{P,\alpha}$-modules and provides an equivalence between the category of $\mathcal{D}_{P,\alpha}$-modules and that of $\mathcal{B}$-modules. We do not use this claim, so its proof is left to the reader.

*Proof of Proposition 7.9.* As we have already mentioned, the claims are obvious on $\mathbb{P}^1 - \mathfrak{D}$. Therefore, it suffices to consider the formal neighborhood of a point $x_i$. Since we concentrate on a single point, we drop the index $i$ to simplify the notation, so $\alpha^{\pm} = \alpha_i^{\pm}$, $z = z_i$, and $n = n_i$.

Let $\mathcal{D}_K := \mathbb{C}((z))\langle \frac{\mathbf{d}}{\mathbf{d}z} \rangle$ be the ring of differential operators on the punctured formal neighborhood of $x_i$. Set

$$\delta := z^n \frac{\mathbf{d}}{\mathbf{d}z}, \quad B := z^{-n}\left(z^n \frac{\mathbf{d} - \alpha^+}{\mathbf{d}z}\right)\left(z^n \frac{\mathbf{d} - \alpha^-}{\mathbf{d}z}\right) \in \mathcal{D}_K$$

and

$$\mathcal{B}_O := \mathbb{C}[[z]]\,\langle \delta, B \rangle \subset \mathcal{D}_K, \qquad \mathcal{D}_O^{\pm} := \mathbb{C}[[z]]\left\langle \frac{\mathbf{d} - \alpha^{\pm}}{\mathbf{d}z} \right\rangle \subset \mathcal{D}_K.$$

Then the restriction of $\mathcal{B}$ to the formal neighborhood of $x_i$ is $\mathcal{B}_O$, the restriction of $\mathcal{D}_{P,\alpha}$ to the formal neighborhoods of $x_i^{\pm}$ is $\mathcal{D}_O^{\pm}$, and the restriction of $R^0\wp_*\mathcal{D}_{P,\alpha}$ (resp. $R^1\wp_*\mathcal{D}_{P,\alpha}$) to the formal neighborhood of $x_i$ equals $\mathcal{D}_O^+ \cap \mathcal{D}_O^-$ (resp. $\mathcal{D}_K/(\mathcal{D}_O^+ + \mathcal{D}_O^-)$). The proposition thus reduces to the following statements:

(1) $\mathcal{B}_O = \mathcal{D}_O^+ \cap \mathcal{D}_O^-$,
(2) $\mathcal{D}_K = \mathcal{D}_O^+ + \mathcal{D}_O^-$,
(3) The set $\{1, \delta, B, B\delta, B^2, B^2\delta, \dots\}$ is a basis of $\mathcal{B}_O$ as of a $\mathbb{C}[[z]]$-module. (Note that the symbol of $\delta$ (resp. the symbol of $B$) is a section of $\mathcal{T}_{\mathbb{P}^1}(-\mathfrak{D})$ (resp. of $\mathcal{T}_{\mathbb{P}^1}^{\otimes 2}(-\mathfrak{D})$).

Set $F := \mathbb{C}[[z]]\langle \delta \rangle \subset \mathcal{D}_K$, and introduce the filtration

$$\cdots \subset zF \subset F \subset z^{-1}F \subset \cdots \subset \mathcal{D}_K.$$

For an element $C \in \mathcal{D}_K$ denote by $\bar{C}$ its image in $\operatorname{gr} \mathcal{D}_K$.

**Lemma 7.11.** (*a*) *This filtration is exhaustive, separated, and compatible with the ring structure.*
(*b*) *If $n > 1$, then the associated graded ring is isomorphic to $\mathbb{C}[\bar{z}, \bar{z}^{-1}, \bar{\delta}]$, that is, to the ring of functions on $\mathbb{A}^1 \times (\mathbb{A}^1 - 0)$.*
(*c*) *For $n = 1$ the associated graded ring is isomorphic to $\mathbb{C}[\bar{z}, \bar{z}^{-1}]\langle \bar{\delta} \rangle/(\bar{\delta}\bar{z} - \bar{z}\bar{\delta} - \bar{z})$, that is, to the ring of differential operators on $\mathbb{A}^1 - 0$.*

*Proof.* (a) Note first that every element of $C \in \mathcal{D}_K$ can be written uniquely as

$$C = \sum_{l \geq 0} f_l(z)\delta^l, \tag{7.5}$$

where $f_l(z) \in \mathbb{C}((z))$. It follows from commutation relation

$$[\delta, z^k] = kz^{k+n-1} \tag{7.6}$$

that $C \in F$ if and only if for all $l$ we have $f_l(z) \in \mathbb{C}[[z]]$. Thus $C \in z^{-k}F$ if and only if for all $l$ we have $z^k f_l(z) \in \mathbb{C}[[z]]$. Hence the filtration is exhaustive and separated.

It follows from commutation relation (7.6) by induction on $l$ that $\delta^l z^k \in z^k F$. Now it is easy to see that the filtration is compatible with the ring structure.

(b) It follows from (7.6) that $\bar{z}$ and $\bar{\delta}$ commute in $\operatorname{gr} \mathcal{D}_k$ if $n > 1$. Thus we get a homomorphism

$$\mathbb{C}[\bar{z}, \bar{z}^{-1}, \bar{\delta}] \to \operatorname{gr} \mathcal{D}_K.$$

Using presentation (7.5), we see that it is bijective.

The proof of (c) is similar to that of (b). □

Denote by $a^\pm$ the leading coefficient of $\alpha^\pm = a^\pm z^{-n}\mathbf{d}z + \dots$, and define the polynomials $q_l^\pm(t)$ for a non-negative integer $l$ by

$$q_l^\pm(t) := \begin{cases} (t - a^\pm)^l & \text{if } n > 1, \\ \prod_{i=0}^{l-1}(t - a^\pm - i) & \text{if } n = 1. \end{cases}$$

**Lemma 7.12.** *(a) The image of* $\left(\frac{\mathbf{d}-\alpha^\pm}{\mathbf{d}z}\right)^l$ *in* $\operatorname{gr}\mathcal{D}_K$ *is* $\bar{z}^{-nl}q_l^\pm(\bar{\delta})$.
*(b) The image of* $B^l$ *in* $\operatorname{gr}\mathcal{D}_K$ *is* $\bar{z}^{-nl}q_l^+(\bar{\delta})q_l^-(\bar{\delta})$.

*Proof.* (a) The image of $\frac{\mathbf{d}-\alpha^\pm}{\mathbf{d}z}$ in $\operatorname{gr}\mathcal{D}_K$ is $\bar{z}^{-n}(\bar{\delta}-a^\pm)$. If $n > 1$, then the statement follows from commutativity of $\operatorname{gr}\mathcal{D}_K$.

If $n = 1$, then we have to move all copies of $\bar{z}^{-1}$ to the left in $(\bar{z}^{-1}(\bar{\delta} - a^\pm))^l$. Now the statement follows from the relation

$$(\bar{\delta} - a)\bar{z}^{-1} = \bar{z}^{-1}(\bar{\delta} - a - 1), \quad a \in \mathbb{C}. \tag{7.7}$$

(b) We have $\bar{B} = \bar{z}^{-n}(\bar{\delta} - a^+)(\bar{\delta} - a^-)$. Now the case $n > 1$ is obvious, the case $n = 1$ again follows from (7.7). □

By Lemma 7.11 any element of $\operatorname{gr}_{-k}\mathcal{D}_K$ can be uniquely written as

$$\bar{z}^{-k}p(\bar{\delta}), \quad p(\bar{\delta}) \in \mathbb{C}[\bar{\delta}].$$

Denote by $\operatorname{gr}\mathcal{D}_O^\pm$ the set of images of all elements of $\mathcal{D}_O^\pm$ in $\operatorname{gr}\mathcal{D}_K$. Define $\operatorname{gr}\mathcal{B}_O$ similarly. Fix $k \in \mathbb{Z}$ and set $l := \lceil \frac{k}{n} \rceil$.

**Lemma 7.13.**

*(a)* $\bar{z}^{-k}p(\bar{\delta}) \in \operatorname{gr}\mathcal{D}_O^\pm$ *if and only if* $k \le 0$ *or* $q_l^\pm(t)|p(t)$;

*(b)* $\bar{z}^{-k}p(\bar{\delta}) \in \operatorname{gr}\mathcal{B}_O$ *if and only if* $k \le 0$ *or* $q_l^+(t)q_l^-(t)|p(t)$.

*Proof.* (a) Consider any element $C \in \mathcal{D}_O^+$, $C \neq 0$. It is easy to see that it can be uniquely written as

$$\sum_{i,j \ge 0} f_{ij}z^j\left(\frac{\mathbf{d}-\alpha^+}{\mathbf{d}z}\right)^i$$

with $f_{ij} \in \mathbb{C}$. Let $k$ be the maximum value of the function $(i,j) \mapsto ni - j$ on the set $\{(i,j) | \, f_{ij} \neq 0\}$. Then Lemma 7.12(a) shows that

$$C \in \sum_{ni-j=k} f_{ij}z^{-k}q_i^+(\delta) + z^{1-k}F.$$

Since elements $\bar{z}^{-k}q_i^+(\bar{\delta})$ form a basis in $\operatorname{gr}_{-k}\mathcal{D}_K$, we see that $C \notin z^{1-k}F$ and

$$\bar{C} = \sum_{ni-j=k} f_{ij}\bar{z}^{-k}q_i^+(\bar{\delta}).$$

Since $j \geq 0$, we see that $i \geq \frac{k}{n}$, so $i \geq l$. Thus if $k > 0$, then $\bar{C} = \bar{z}^{-k}p(\bar{\delta})$, where $p$ is divisible by $q_l^+$.

Conversely, given a polynomial $p$ divisible by $q_l^+$ (or any polynomial if $k \leq 0$), we can write $p = \sum_{i\geq l} f_i q_i^+$ with $f_i \in \mathbb{C}$. Set

$$C = \sum_i f_i z^{ni-k}\left(\frac{\mathbf{d} - \alpha^+}{\mathbf{d}z}\right)^i.$$

Then $C \in \mathcal{D}_O^+$ and $\bar{C} = \bar{z}^{-k}p(\bar{\delta})$. The case of $\mathcal{D}_O^-$ is completely similar.

(b) Consider $C \in \mathcal{B}_O$ with $\bar{C} = \bar{z}^{-k}p(\bar{\delta})$. It is easy to see that $\mathcal{B}_O \subset \mathcal{D}_O^+ \cap \mathcal{D}_O^-$. Thus it follows from part (a) that $q_l^{\pm}(t)$ divides $p(t)$. Thus $q_l^+(t)q_l^-(t)$ divides $p(t)$, since $q_l^-(t)$ and $q_l^+(t)$ are coprime. Finally, assume that $q_l^+(t)q_l^-(t)$ divides $p(t)$, we can write

$$p(t) = \sum_{i\geq l} f_i q_i^+(t)q_i^-(t) + \sum_{i\geq l} g_i t q_i^+(t)q_i^-(t), \qquad f_i, g_i \in \mathbb{C}.$$

Set

$$C = \sum_i f_i z^{ni-k}B^i + \sum_i g_i z^{ni-k}B^i\delta.$$

Clearly, $\bar{C} = \bar{z}^{-k}p(\bar{\delta})$. □

We now see that the identities (1)–(2) hold in the associated graded ring of $\mathcal{D}_K$, and hence also in $\mathcal{D}_K$ itself. The proof of part (b) of the lemma shows that every element of $\operatorname{gr}\mathcal{B}_O \cap \operatorname{gr}_{-k}\mathcal{D}_K$ can be uniquely written as

$$\sum_{i\geq k/n} f_i\bar{z}^{ni-k}\bar{B}^i + \sum_{i\geq k/n} g_i\bar{z}^{ni-k}\bar{B}^i\bar{\delta}$$

with $f_i, g_i \in \mathbb{C}$, and (3) follows. The proof of Proposition 7.9 is complete. □

*Proof of Proposition 7.3.* Our first goal is to reformulate the proposition as a statement about the cokernel of a map between sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$.

Note first of all that any $p_1^\bullet\mathcal{D}_{P,\alpha} \circledast p_2^\bullet\mathcal{D}_{P,-\alpha}$-module $\mathcal{G}$ on $P \times P$ is, in particular, a $p_1^{-1}\mathcal{D}_{P,\alpha}$-module. Therefore, $(\wp\times\wp)_*\mathcal{G}$ has a natural structure of a $p_1^{-1}\mathcal{B}$-module coming from the isomorphism of Proposition 7.9(a). (There is also a commuting structure of a $p_2^{-1}\mathcal{B}_{-\alpha}$-module that we do not use.)

Thus

$$(\wp\times\wp)_*H^1(\varphi) : (\wp\times\wp)_*\delta_\Delta \to (\wp\times\wp)_*R^1p_{12,*}\mathcal{F}_P \tag{7.8}$$

is a map of $p_1^{-1}\mathcal{B}$-modules.

Consider $\delta_\Delta$ as a $p_1^{-1}\mathcal{D}_{P,\alpha}$-module. It is isomorphic to $\iota_{\Delta,*}(\mathcal{D}_{P,\alpha}\otimes_{O_P}\mathcal{T}_P)$, where $\iota_{\Delta,*}$ is the $O$-module pushforward. By the projection formula, Proposition 7.9 gives an isomorphism in the derived category of $p_1^{-1}\mathcal{B}$-modules

$$R(\wp\times\wp)_*\delta_\Delta = \iota_{\Delta,*}(\mathcal{B}\otimes_{O_{\mathbb{P}^1}}\mathcal{T}_{\mathbb{P}^1}). \tag{7.9}$$

Using this and (7.1), we re-write (7.8) as

$$(\wp\times\wp)_*H^1(\varphi) : \iota_{\Delta,*}(\mathcal{B}\otimes_{O_{\mathbb{P}^1}}\mathcal{T}_{\mathbb{P}^1}) \to R^1p_{12,*}\mathcal{F}_{\mathbb{P}^1}. \tag{7.10}$$

As was explained in the proof of Theorem 4, $H^1(\varphi)$ is injective. Also, $R^1(\wp \times \wp)_*\delta_\Delta = 0$ by (7.9). We now see that

$$(\wp \times \wp)_* \operatorname{Coker}(H^1(\varphi)) = \operatorname{Coker}((\wp \times \wp)_* H^1(\varphi)).$$

Thus it remains to prove that the cokernel of (7.10) is coherent. Note that (7.10) is an injective maps between $p_1^{-1}\mathcal{B}$-modules.

Now recall that $\mathcal{F}_{\mathbb{P}^1}$ naturally extends to a vector bundle $\overline{\mathcal{F}} := p_{13}^*\xi \otimes p_{23}^*\xi^\vee$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \overline{\mathcal{M}}$, which provides the filtration (7.2). By Proposition 6.10 we obtain an induced filtration

$$R^1 p_{12,*}\mathcal{F}_0 \subset \cdots \subset R^1 p_{12,*}\mathcal{F}_k \subset \cdots \subset R^1 p_{12,*}\mathcal{F}_\infty = R^1 p_{12,*}\mathcal{F}_{\mathbb{P}^1} \tag{7.11}$$

of $R^1 p_{12,*}\mathcal{F}_{\mathbb{P}^1}$ by coherent sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$.

**Lemma 7.14.** *There is $l \in \mathbb{Z}$ such that for $k \gg 0$ the image of $\iota_{\Delta,*}(\mathcal{B}^{\leq k} \otimes_{O_{\mathbb{P}^1}} \mathcal{T}_{\mathbb{P}^1})$ under (7.10) is contained in $R^1 p_{12,*}\mathcal{F}_{k+l}$ and such that the induced map*

$$\iota_{\Delta,*}((\mathcal{B}^{\leq k+1}/\mathcal{B}^{\leq k}) \otimes_{O_{\mathbb{P}^1}} \mathcal{T}_{\mathbb{P}^1}) \to R^1 p_{12,*}\mathcal{F}_{k+l+1}/R^1 p_{12,*}\mathcal{F}_{k+l}. \tag{7.12}$$

*is an isomorphism.*

*Proof.* By construction, the filtration (7.2) agrees with the filtration on $\mathcal{B}$, so that $(p_1^{-1}\mathcal{B}^{\leq l})\mathcal{F}_k \subset \mathcal{F}_{k+l}$. Therefore, the filtration (7.11) also agrees with the filtration on $\mathcal{B}$. Using Remark 7.10(ii) and Proposition 6.10, we see that for $k \gg 0$

(7.13)

$$\iota_{\Delta,*}((\mathcal{B}^{\leq k}/\mathcal{B}^{\leq k-1}) \otimes_{O_{\mathbb{P}^1}} \mathcal{T}_{\mathbb{P}^1}) \simeq R^1 p_{12,*}\mathcal{F}_k/R^1 p_{12,*}\mathcal{F}_{k-1} \simeq \begin{cases} \iota_{\Delta,*}\mathcal{T}_{\mathbb{P}^1}^{\otimes 2}(-\mathfrak{D}) & \text{if } k \text{ is odd,} \\ \iota_{\Delta,*}\mathcal{T}_{\mathbb{P}^1} & \text{if } k \text{ is even.} \end{cases}$$

For each $k$, let $l(k)$ be the smallest index such that the image of $\iota_{\Delta,*}(\mathcal{B}^{\leq k} \otimes_{O_{\mathbb{P}^1}} \mathcal{T}_{\mathbb{P}^1})$ is contained in $R^1 p_{12,*}\mathcal{F}_{k+l(k)}$. Since the filtration (7.11) agrees with the filtration on $\mathcal{B}$, we see that $l(k+1) \leq l(k)$ for all $k$. Also, injectivity of (7.10) implies that

$$0 \leq \operatorname{rk} \iota_\Delta^*(R^1 p_{12,*}\mathcal{F}_{k+l(k)}) - \operatorname{rk}(\mathcal{B}^{\leq k} \otimes_{O_{\mathbb{P}^1}} \mathcal{T}_{\mathbb{P}^1}) = (k + l(k) + \operatorname{rk} \iota_\Delta^*(R^1 p_{12,*}\mathcal{F}_0)) - k,$$

and therefore $l(k) \geq -\operatorname{rk} \iota_\Delta^*(R^1 p_{12,*}\mathcal{F}_0)$. Thus $l(k)$ stabilizes as $k \to \infty$; set $l := \lim l(k)$.

By the choice of $l$, the map (7.12) is non-zero for $k \gg 0$. Note that such non-zero map does not exist if $k$ and $l$ are odd. Therefore, $l$ must be even. We now see that for $k \gg 0$, the map (7.12) is a non-zero morphism between isomorphic line bundles on $\Delta$. This implies (7.12) is an isomorphism. □

It follows from the lemma that $R^1 p_{12,*}\mathcal{F}_k$ maps surjectively onto $\operatorname{Coker}(\wp \times \wp)_* H^1(\varphi)$ for $k \gg 0$. This completes the proof of the proposition and of Theorem 4. □

## 8. Second orthogonality relation.

In this section we prove Theorem 5. The proof is similar to [Ari2] but we want to give some details.

We need to calculate $Rp_{12,*}\,\mathbb{DR}(\mathcal{F}_\mathcal{M})$, where $p_{12} : \mathcal{M} \times \mathcal{M} \times P \to \mathcal{M} \times \mathcal{M}$. Our first goal is to reduce the problem to a calculation on $\mathcal{M} \times \mathcal{M} \times \mathbb{P}^1$. Recall that $\xi$ is the universal bundle on $\mathcal{M} \times \mathbb{P}^1$, set $\xi_{12} := \mathcal{H}om(p_{23}^*\xi, p_{13}^*\xi)$. We have a connection along $\mathbb{P}^1$

$$\operatorname{ad}\nabla : \xi_{12} \to \xi_{12} \otimes p_3^*\Omega_{\mathbb{P}^1}(\mathfrak{D}).$$

Its polar part is a well-defined $O$-linear map

$$\xi_{12}|_{\mathcal{M}\times\mathcal{M}\times\mathfrak{D}} \to (\xi_{12}\otimes p_3^*\Omega_{\mathbb{P}^1}(\mathfrak{D}))|_{\mathcal{M}\times\mathcal{M}\times\mathfrak{D}}.$$

Let $\eta_{12}$ be the image of this map. Denote by $\tilde{\xi}_{12}$ the modification of $\xi_{12}\otimes p_3^*\Omega_{\mathbb{P}^1}$ whose sheaf of sections is

$$\{s\in \xi_{12}\otimes p_3^*\Omega_{\mathbb{P}^1}(\mathfrak{D})|_{\mathcal{M}\times\mathcal{M}\times\mathfrak{D}} : s|_{\mathcal{M}\times\mathcal{M}\times\mathfrak{D}}\in\eta_{12}\}.$$

As in [Ari2, Lemmas 12, 13] one proves that

$$Rp_{12,*}\,\mathbb{DR}(\mathcal{F}_{\mathcal{M}}) = Rp_{12,*}(\xi_{12}\xrightarrow{\mathrm{ad}\,\nabla}\tilde{\xi}_{12}).$$

The next step is to calculate the restriction of the above to a fiber over a point. So consider a closed point $x\in\mathcal{M}\times\mathcal{M}$ and let $\mathcal{F}^\bullet$ be the restriction of $\xi_{12}\xrightarrow{\mathrm{ad}\,\nabla}\tilde{\xi}_{12}$ to $x$.

**Proposition 8.1.** *(a) If $x\notin\operatorname{diag}(\mathcal{M})$, then $\mathbb{H}^i(\mathcal{F}^\bullet)=0$ for any $i$;*
*(b) If $x\in\operatorname{diag}(\mathcal{M})$, then*

$$\dim\mathbb{H}^i(\mathcal{F}^\bullet)=\begin{cases}1 & \text{if } i=0,2,\\ 2 & \text{if } i=1,\\ 0 & \text{otherwise.}\end{cases}$$

*(c) Suppose $x=((L,\nabla),(L,\nabla))\in\mathcal{M}\times\mathcal{M}$. Consider the map of complexes*

$$(O_{\mathbb{P}^1}\xrightarrow{\mathbf{d}}\Omega_{\mathbb{P}^1})\hookrightarrow\mathcal{F}^\bullet$$

*induced by $O_{\mathbb{P}^1}\to\xi_{12}|_x : f\mapsto f\,\mathrm{id}_L$. Then the induced map*

$$H^i_{DR}(\mathbb{P}^1,\mathbb{C}):=\mathbb{H}^i(\mathbb{P}^1,O_{\mathbb{P}^1}\xrightarrow{\mathbf{d}}\Omega_{\mathbb{P}^1})\to\mathbb{H}^i(\mathcal{F}^\bullet)$$

*is an isomorphism for $i=0,2$.*

The proof is analogous to the proof of [Ari2, Proposition 10]: one uses irreducibility, duality, and Euler characteristic.

As in [Ari2, Lemma 14] the duality gives the following

**Lemma 8.2.** *Let $S$ be a locally Noetherian scheme, $\iota: S\to\mathcal{M}\times\mathcal{M}$. Set*

$$\mathcal{F}_{(S)}:=Rp_{1,*}((\iota\times\mathrm{id}_{\mathbb{P}^1})^*(\xi_{12}\xrightarrow{\mathrm{ad}\,\nabla}\tilde{\xi}_{12}))$$

*(here $p_1: S\times\mathbb{P}^1\to S$). Then $\mathcal{H}om(H^2(\mathcal{F}_{(S)}),O_S)$ is isomorphic to a subsheaf of $H^0(\mathcal{F}_{(S)})$.*

Next, $\operatorname{diag}:\mathcal{M}\to\mathcal{M}\times\mathcal{M}$ is a $\mathbf{G_m}$-torsor over $\operatorname{diag}(\mathcal{M})$ (cf. Remark 2.7). Denote by Hom the corresponding line bundle. Note that the fiber of Hom over $((L_1,\nabla_1),(L_2,\nabla_2))$ is $\{A\in\operatorname{Hom}_{O_{\mathbb{P}^1}}(L_1,L_2) : A\nabla_1=\nabla_2A\}$. Now the following corollary of Proposition 8.1 is obvious.

**Corollary 8.3.** *(a) $Rp_{12,*}(\xi_{12}\xrightarrow{\mathrm{ad}\,\nabla}\tilde{\xi}_{12})$ vanishes if restricted to $\mathcal{M}\times\mathcal{M}-\operatorname{diag}(\mathcal{M})$.*
*(b) The map*

$$p_{12}^*\operatorname{Hom}\otimes p_3^*(O_{\mathbb{P}^1}\xrightarrow{\mathbf{d}}\Omega_{\mathbb{P}^1})\to(\xi_{12}\xrightarrow{\mathrm{ad}\,\nabla}\tilde{\xi}_{12})|_{\operatorname{diag}(\mathcal{M})}$$

*induces an isomorphism*

$$\operatorname{Hom}=\operatorname{Hom}\otimes\mathbb{H}^2(\mathbb{P}^1,(O_{\mathbb{P}^1}\xrightarrow{\mathbf{d}}\Omega_{\mathbb{P}^1}))\to R^2p_{12,*}\left((\xi_{12}\xrightarrow{\mathrm{ad}\,\nabla}\tilde{\xi}_{12})|_{\operatorname{diag}(\mathcal{M})}\right).$$

Let us use the following observation (cf. [Ari2, Lemma 15] and [Mum, Lemma in §13]).

**Lemma 8.4.** *Let $Z$ be a locally Noetherian scheme, $Y \subset Z$ a closed subscheme that is locally a complete intersection of pure codimension $n$. Denote by $\iota : Y \hookrightarrow Z$ the natural embedding.*

(*a*) *Let $\mathcal{F}$ be a quasi-coherent sheaf on $Z$ such that $\mathcal{F}|_{Z-Y} = 0$, $L_n\iota^*\mathcal{F} = 0$. Then $\mathcal{F} = 0$.*

(*b*) *Let $\mathcal{F}^\bullet = (\mathcal{F}^0 \to \mathcal{F}^1 \to \dots)$ be a complex of flat quasi-coherent sheaves on $Z$ such that $H^i(\mathcal{F}^\bullet)|_{Z-Y} = 0$ for all $i < n$. Then $H^i(\mathcal{F}^\bullet) = 0$ for $i < n$.*

*Proof of Theorem 5.* Clearly, $\operatorname{diag}(\mathcal{M}) = M \times B(\mathbf{G_m} \times \mathbf{G_m})$ is a complete intersection in $\mathcal{M}\times\mathcal{M} = M\times M\times B(\mathbf{G_m}\times\mathbf{G_m})$. Thus Lemma 8.4(b) and Corollary 8.3(a) imply that $R^ip_{12,*}(\xi_{12} \xrightarrow{\operatorname{ad}\nabla} \tilde{\xi}_{12}) = 0$ for $i \neq 2$. Set $\mathcal{F}^{(2)} := R^2p_{12,*}(\xi_{12} \xrightarrow{\operatorname{ad}\nabla} \tilde{\xi}_{12})$. Corollary 8.3(b) implies that $\mathrm{Hom} = \mathcal{F}^{(2)}|_{\operatorname{diag}(\mathcal{M})}$. It is easy to see that Hom, viewed as a sheaf on $\mathcal{M} \times \mathcal{M}$, is equal to $(\operatorname{diag}_* O_{\mathcal{M}})^\psi$, where $\psi$ is the character of $\mathbf{G_m} \times \mathbf{G_m}$ given by $(t_1, t_2) \mapsto t_1/t_2$ (because a 1-dimensional vector space $E$ can be identified with weight $-1$ functions on $E - \{0\}$).

To complete the proof, it remains to check that $\mathcal{F}^{(2)}$ is concentrated (scheme-theoretically) on $\operatorname{diag}(\mathcal{M})$. Assume for a contradiction that it is not the case. Note that $\mathcal{F}^{(2)}$ is coherent and concentrated set-theoretically on $\operatorname{diag}(\mathcal{M})$.

**Lemma 8.5.** *Let $Z$ be a locally Noetherian scheme, $Y \subset Z$ be a closed subscheme. Let $\mathcal{G}$ be a coherent sheaf on $Z$ concentrated set-theoretically but not scheme-theoretically on $Y$. Then there is a local Artinian scheme $S$ and an $S$-point of $Z$ such that $S^{red}$ factors through $Y$ and such that the restriction of $\mathcal{G}$ to $S$ is not concentrated on the scheme-theoretic preimage of $Y$.*

We see that there is an $S$-point of $\mathcal{M} \times \mathcal{M}$ such that the restriction of $\mathcal{F}^{(2)}$ to this point is not concentrated on the preimage $S'$ of the diagonal. Let $\mathcal{F}_{(S)}$ be as in Lemma 8.2; using base change we see that $H^2(\mathcal{F}_{(S)})$ is not concentrated on $S'$. The duality for Artinian rings shows that $\mathcal{H}om(H^2(\mathcal{F}_{(S)}), O_S)$ is not concentrated on $S'$ either. But then Lemma 8.2 gives a contradiction, since $H^0(\mathcal{F}_{(S)})$ is easily seen to be concentrated on $S'$. $\square$

*Proof of Lemma 8.5.* We can assume that $Z = \operatorname{Spec} A$, $Y = \operatorname{Spec} A/\mathfrak{a}$, $\mathcal{G}$ corresponds to an $A$-module $M$; by assumption $\mathfrak{a}M \neq 0$ but $\mathfrak{a}^nM=0$ for $n$ big enough. Consider a maximal ideal $\mathfrak{m}$ such that $(\mathfrak{a}M)_\mathfrak{m} \neq 0$. It follows that $\mathfrak{m} \supset \mathfrak{a}$. By Nakayama's Lemma $\cap_n \mathfrak{m}^nM_\mathfrak{m} = 0$ and we can choose $n$ such that $\mathfrak{a}M \not\subset \mathfrak{m}^nM$. We can take $S = \operatorname{Spec}(A/\mathfrak{m}^n)$. $\square$

## 9. Relation to the Langlands correspondence

In this section we prove Theorem 3. Let us present the main steps. Recall that $\mathfrak{D} = \sum n_ix_i$. Set

$$\beta_i^+ := \alpha_i^+ + \frac{n_i\lambda}{2}\frac{\mathbf{d}z_i}{z_i}, \qquad \beta_i^- := \alpha_i^- - \frac{n_i\lambda}{2}\frac{\mathbf{d}z_i}{z_i},$$

where $\lambda := \sum_i \operatorname{res}\alpha_i^-$, $z_i$ is a local parameter at $x_i$. Note that the polar parts $\beta_i^\pm$ do not depend on the choice of $z_i$. Denote now the moduli space $\mathcal{M}$ defined in §2.1 by $\mathcal{M}_\alpha$ to make the choice of formal types explicit. For a sheaf $\mathcal{R}$ of rings we denote

by $\mathcal{R}$-mod the category of left $\mathcal{R}$-modules, by $\mathcal{D}^b(\mathcal{R})$ the bounded derived category of $\mathcal{R}$-mod. We shall prove first that

$$\mathcal{D}_{\overline{\mathcal{B}un}(-1),\alpha}\text{-mod} = \mathcal{D}_{\mathcal{B}un(-1),\alpha}\text{-mod} \simeq \mathcal{D}_{P,\beta}\text{-mod}.$$

It remains to prove the following equivalences

$$\mathcal{D}^b(\mathcal{D}_{P,\beta}) \simeq \mathcal{D}^b(\mathcal{M}_\beta)^{(-1)} \simeq \mathcal{D}^b(\mathcal{M}_\alpha)^{(-1)}.$$

Note that if $\alpha_i^\pm$ satisfy the conditions of §2.1, then $\beta_i^\pm$ satisfy these conditions as well ((c) and (d) can be checked case by case). Thus the first equivalence follows from Theorem 2. For the last equivalence we shall prove that $\mathcal{M}_\alpha \simeq \mathcal{M}_\beta$.

It is well known that the definition of the derived category of $\mathcal{D}$-modules on a stack requires some caution. In this paper, we ignore the difficulty and use the naive definition: the derived category of $\mathcal{D}_{\overline{\mathcal{B}un}(-1),\alpha}$-modules is simply the derived category of the abelian category of $\mathcal{D}_{\overline{\mathcal{B}un}(-1),\alpha}$-modules.

9.1. **Twisted $\mathcal{D}$-modules on algebraic stacks.** Let us summarize the properties of modules over TDO rings on algebraic stacks. We make no attempt to work in most general settings, and consider only smooth stacks, and only twists induced by torsors over an algebraic group. This case is enough for our purposes.

Let $G$ be an algebraic group with Lie algebra $\mathrm{Lie}(G)$. Fix a $G$-invariant functional $\theta : \mathrm{Lie}(G) \to \mathbb{C}$. First, consider twisted differential operators on a variety.

Let $X$ be a smooth variety and let $p : T \to X$ be a $G$-torsor on $X$. These data determine a TDO ring $\mathcal{D}_{X,T,\theta}$ on $X$, which is obtained by non-commutative reduction of the sheaf of differential operators $\mathcal{D}_T$ on $T$.

Namely, every $\xi \in \mathrm{Lie}(G)$ gives a first order differential operator $a(\xi) - \theta(\xi) \in \mathcal{D}_T$, where the vector field $a(\xi)$ on $T$ is the action of $\xi$. Let $I$ be the ideal in $p_*\mathcal{D}_T$ generated by these differential operators. It is easy to see that this ideal is $G$-invariant, and we set $\mathcal{D}_{X,T,\theta} := (p_*\mathcal{D}_T/I)^G$.

The category of quasi-coherent $\mathcal{D}_{X,T,\theta}$-modules can be described using a twisted strong equivariance condition. Let $M$ be a $\mathcal{D}_T$-module equipped with a weak $G$-equivariant structure (that is, $M$ is $G$-equivariant as a quasi-coherent sheaf, and the structure of a $\mathcal{D}_T$-module is $G$-equivariant). We say that $M$ is *strongly equivariant with twist* $\theta$ if the action of $\xi \in \mathrm{Lie}(G)$ on $M$ induced by the $G$-equivariant structure is given by $a(\xi) - \theta(\xi)$.

*Remark* 9.1. The sheaves of twisted differential operators have been introduced in [BB1]. The correspondence between $\mathcal{D}_{X,T,\theta}$-modules and twisted strongly equivariant modules is a particular case of the formalism of Harish-Chandra algebras from [BB1, §1.8].

Let now $\mathcal{X}$ be an algebraic stack, and let $T \to \mathcal{X}$ be a $G$-torsor on $\mathcal{X}$. Every smooth morphism $\alpha : X \to \mathcal{X}$ from a variety $X$ induces a $G$-torsor $\alpha^*T$ on $X$, and we obtain the TDO ring $\mathcal{D}_{X,\alpha^*T,\theta}$. Such TDO rings form a $\mathcal{D}$-algebra on $\mathcal{X}$ in the sense of [BB1]. We denote this $\mathcal{D}$-algebra by $\mathcal{D}_{\mathcal{X},T,\theta}$. Note that $\mathcal{D}_{\mathcal{X},T,\theta}$ is not a sheaf of algebras on $\mathcal{X}$.

By definition, a $\mathcal{D}_{\mathcal{X},T,\theta}$-module $M$ is given by specifying a $\mathcal{D}_{X,\alpha^*T,\theta}$-module $M_\alpha$ for every smooth morphism $\alpha : X \to \mathcal{X}$ and an isomorphism of $\mathcal{D}_{Y,(\alpha\circ f)^*T,\theta}$-modules $f^*M_\alpha \simeq M_{\alpha\circ f}$ for every smooth map $f : Y \to X$ of algebraic varieties; the isomorphisms must be compatible with composition of morphisms $f$. Note in particular that $M$ is a quasi-coherent sheaf on $\mathcal{X}$.

*Example* 9.2. Let $\mathcal{X} := B(H)$ be the classifying stack of an algebraic group $H$. Set $X = \operatorname{Spec}\mathbb{C}$. The natural map $\alpha : \operatorname{Spec}\mathbb{C} \to \mathcal{X}$ is an $H$-torsor (and, in particular, a presentation). For any $G$-torsor $T$ on $\mathcal{X}$, the pullback $\alpha^*T$ is isomorphic to the trivial torsor $G \to \operatorname{Spec}\mathbb{C}$. Fix a trivialization $\alpha^*T \simeq G$. The group $H$ acts on $\alpha^*T = G$; this is the right action for a homomorphism $\psi : H \to G$. In other words, $T$ is the descent of $G \to \operatorname{Spec}\mathbb{C}$, and $\psi$ provides the descent datum.

Let $M$ be a $\mathcal{D}_{\mathcal{X},T,\theta}$-module. It is easy to see that the TDO ring $\mathcal{D}_{\operatorname{Spec}\mathbb{C},G,\theta}$ is just the field of complex numbers, so the $\mathcal{D}_{\operatorname{Spec}\mathbb{C},G,\theta}$-module $\alpha^*M$ is a vector space $V$. Let us view $\alpha^*M$ as a strongly $G$-equivariant $\mathcal{D}_{\alpha^*T}$-module with twist $\theta$. It corresponds to the free $O_G$-module $V \otimes_{\mathbb{C}} O_G$ with the obvious $G$-equivariant structure. The action of $\mathcal{D}_{\alpha^*T}$ is uniquely determined by the twisted strong equivariance condition. On the other hand, $\alpha^*M$ also carries a structure of a strongly $H$-equivariant $\mathcal{D}$-module; this structure is essentially the descent data for $M$. If $V \neq 0$, then such structure is provided by a scalar representation of $H$ on $V$ whose derivative is $-\theta \circ \mathbf{d}\psi$.

In particular, suppose that the character $\theta \circ \mathbf{d}\psi : \operatorname{Lie}(H) \to \mathbb{C}$ does not integrate to a representation $H_0 \to \mathbf{G_m}$, where $H_0 \subset H$ is the identity component. Then $V = 0$ and therefore the only $\mathcal{D}_{\mathcal{X},T,\theta}$-module is the zero module.

9.2. **Step 1:** $\mathcal{D}_{\overline{\mathcal{B}un}(-1),\alpha}-\mathrm{mod} = \mathcal{D}_{\mathcal{B}un(-1),\alpha}-\mathrm{mod}$**.** In this section we shall prove

**Proposition 9.3.** *Assume that $(L,\eta) \in \overline{\mathcal{B}un}(-1)$ does not correspond to a connection $(L,\nabla) \in \mathcal{M}$ in the sense of §4.3. Then the restriction of any $\mathcal{D}_{\overline{\mathcal{B}un}(-1),\alpha}$-module $\xi$ to $(L,\eta)$ is zero.*

This proposition and Proposition 4.8(b) imply that the restriction of every $\mathcal{D}_{\overline{\mathcal{B}un}(-1),\alpha}$-module to a point $(L,\eta)$ such that $(L,\eta) \notin \mathcal{B}un(-1)$ is zero. This is valid for not necessarily closed points, and the step follows.

*Proof of proposition.* The proof is based on Corollary 4.4 and Example 9.2.

Consider Example 9.2 with $H = \operatorname{Aut}(L,\eta)$. Recall from §2.4 that the twist is given by the torsor
$$T = \eta_{univ} \times_{\overline{\mathcal{B}un}(-1)} \eta'_{univ}$$
over $G = \mathbb{C}[\mathfrak{D}]^\times \times \mathbb{C}[\mathfrak{D}]^\times$ and the character $\theta = (\alpha^+, \alpha^-)$. One easily checks that $\psi : \operatorname{Aut}(L,\eta) \to \mathbb{C}[\mathfrak{D}]^\times \times \mathbb{C}[\mathfrak{D}]^\times$ is given by the action of automorphisms on $\eta$ and $(L|_{\mathfrak{D}})/\eta$. Thus, in the notation of Corollary 4.4, $\mathbf{d}\psi : A \mapsto (A_+, A_-)$ and
$$\theta \circ \mathbf{d}\psi : A \mapsto \operatorname{res}(A_+\alpha_+) + \operatorname{res}(A_-\alpha_-). \tag{9.1}$$
Now assume that $(L,\nabla)$ does not correspond to any connection; our goal is to prove that $\theta \circ \mathbf{d}\psi$ does not integrate to a character of the identity component of $\operatorname{Aut}(L,\eta)$. It follows from Corollary 4.4 that there is $A \in \operatorname{End}(L,\eta)$ such that
$$\operatorname{res}(A_+\alpha_+) + \operatorname{res}(A_-\alpha_-) + \langle A, b(L)\rangle \neq 0.$$
It is enough to consider two cases: $A$ is nilpotent, and $A$ is semisimple. In the first case it follows from (9.1) and (4.1) that $\theta \circ \mathbf{d}\psi(A) \neq 0$ and $\theta \circ \mathbf{d}\psi$ cannot integrate to a character $H_0 \to \mathbf{G_m}$, so we are done.

Let $A$ be semisimple. It follows from (4.1) and condition (b) of §2.1 that
$$\operatorname{res}((\mathrm{id}_L)_+\alpha_+) + \operatorname{res}((\mathrm{id}_L)_-\alpha_-) + \langle \mathrm{id}_L, b(L)\rangle = 0.$$
Thus $A$ is not scalar. Then $(L,\eta)$ decomposes with respect to the eigenvalues of $A$ as $L = L_1 \oplus L_2$ (and for every $i$ we have $\eta|_{n_i x_i} = L_1|_{n_i x_i}$ or $\eta|_{n_i x_i} = L_2|_{n_i x_i}$).

Let $A'$ be the endomorphism of $(L,\eta)$ that is zero on $L_1$ and the identity on $L_2$. We see that

$$\operatorname{res}(A'_+\alpha_+) + \operatorname{res}(A'_-\alpha_-) = \sum_i \operatorname{res}\alpha_i^{\pm} \notin \mathbb{Z}$$

by condition (c) of 2.1. Again, $\theta \circ \mathbf{d}\psi$ does not integrate and we are done. □

9.3. **Step 2: $\mathcal{D}_{\mathcal{B}un(-1),\alpha}$ − mod $\simeq \mathcal{D}_{P,\beta}$ − mod.** Recall that $\mathcal{B}un(-1) = P/\mathbf{G_m}$, where $\mathbf{G_m}$ acts trivially.

Let $\pi : P \to \mathcal{B}un(-1)$ be the projection. It follows from the definition of a (strongly) equivariant $\mathcal{D}$-module that $\mathcal{D}_{\mathcal{B}un(-1),\alpha}$ − mod $\simeq \pi^\bullet\mathcal{D}_{\mathcal{B}un(-1),\alpha}$ − mod. So all we have to check is that $\pi^\bullet\mathcal{D}_{\mathcal{B}un(-1),\alpha} = \mathcal{D}_{P,\beta}$.

By Proposition 4.10 we have $\mathcal{B}un(-1) = P/\mathbf{G_m}$, where $P$ is glued from two copies of $\mathbb{P}^1$, which we denote now by $\mathbb{P}^1_+$ and $\mathbb{P}^1_-$ (so that $x_i^- \in \mathbb{P}^1_-$). We saw that $P$ can be viewed as the moduli space of triples $(L,\eta,O_{\mathbb{P}^1}(-2) \hookrightarrow L_\eta)$ (cf. Remark 4.11). We shall be using the notation from the proof of Proposition 4.10. Let $\rho : \tilde{P} \to P$ be the projection. We assume that $\rho^{-1}(\mathbb{P}^1_+)$ is given by $p' \neq 0$, while $\rho^{-1}(\mathbb{P}^1_-)$ is given by $p \neq 0$.

**Lemma 9.4.** *For all $i$ there exists a unique $(L,\eta) \in \mathcal{B}un(-1)$ such that $\eta_{x_i} = (O_{\mathbb{P}^1})_{x_i} \subset L_{x_i}$ and under the above description of $\mathcal{B}un(-1)$ this point corresponds to $x_i^-$.*

*Proof.* Consider the composition $O_{\mathbb{P}^1}(-\mathfrak{D}) \hookrightarrow L(-\mathfrak{D}) \hookrightarrow L_\eta$. Clearly, $\eta_{x_i} = (O_{\mathbb{P}^1})_{x_i}$ if and only if this composition is zero at $x_i$ . This happens if and only if the rank of $\varphi' : O_{\mathbb{P}^1}(-\mathfrak{D}) \oplus O_{\mathbb{P}^1}(-2) \to L_\eta$ drops at $x_i$ with the kernel $O_{\mathbb{P}^1}(-\mathfrak{D})_{x_i}$. This is in turn equivalent to $q = x_i$, $p' = 0$. □

Let $\delta$ be the line bundle on $\mathcal{B}un(-1)$ whose fiber at $(L,\eta)$ is $\det \mathrm{R}\Gamma(\mathbb{P}^1, L)$. Let $\delta'$ be the pullback of $\delta$ to $P$. Fix $\infty \in \mathbb{P}^1 - \mathfrak{D}$.

**Lemma 9.5.** $\delta' \simeq O_P(2(\infty) - \sum n_i x_i^-)$.

*Proof.* Let $t \in \mathbf{G_m}$ act on $\tilde{P}$ by

$$t \cdot (p,p',q) = (p/t, tp', q). \tag{9.2}$$

This action gives rise to a $\mathbf{G_m}$-torsor $\tilde{P} \to P$. We claim that the corresponding line bundle is $\delta'$.

Indeed, consider the cartesian diagram

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\rho} & P \\ \downarrow & & \downarrow \\ P & \longrightarrow & \mathcal{B}un(-1). \end{array}$$

Here the left hand arrow is the torsor described above. The top arrow corresponds to forgetting the embedding $O_{\mathbb{P}^1}(-2) \to L_\eta$. Thus $P$ on the right parameterizes parabolic bundles with embeddings $O_{\mathbb{P}^1} \to L$. However, such an embedding is the same as a non-zero element of $\det \mathrm{R}\Gamma(\mathbb{P}^1, L) = H^0(\mathbb{P}^1, L)$. Hence the torsor on the right is the one corresponding to $\delta$, and the torsor on the left is the one corresponding to $\delta'$.

We have

$$\rho^{-1}(\mathbb{P}^1_+) = \{(f(q)/p', p', q) \in \tilde{P}\},$$

where $p' \neq 0$ is in the fiber of $O_{\mathbb{P}^1}(2)$ over $q$. Thus $\rho^{-1}(\mathbb{P}^1_+)$ is the total space of $O_{\mathbb{P}^1}(2)$ with the zero section removed, and the action (9.2) is the standard one. Hence $\delta'|_{\mathbb{P}^1_+} = O_{\mathbb{P}^1}(2)$.

Further,
$$\rho^{-1}(\mathbb{P}^1_-) = \{(p, f(q)/p, q) \in \tilde{P}\}$$
is also the total space of $O_{\mathbb{P}^1}(2)$ with the zero section removed but the action (9.2) is the inverse one, so the total space of the corresponding line bundle is obtained by compactifying at infinity and $\delta'|_{\mathbb{P}^1_-} = O_{\mathbb{P}^1}(-2)$. We also see that if a meromorphic section $s$ of $\delta'$ has order $m_i$ at $x_i^+$, then it has order $m_i - n_i$ at $x_i^-$.

Let $s$ be a section of $O_{\mathbb{P}^1}(2) = \delta'|_{\mathbb{P}^1_+}$ with a double zero at $\infty$. We view it as a meromorphic section of $\delta'$. It has no other zeroes on $\mathbb{P}^1_+$, and, by the previous remark, it has a pole of order $n_i$ at $x_i^-$. Thus the divisor of $s$ is $2(\infty) - \sum n_i x_i^-$. □

Denote by $\Delta_-$ and $\Delta_+$ the graphs of the immersions $\mathfrak{D} \hookrightarrow \mathbb{P}^1_- \hookrightarrow P$ and $\mathfrak{D} \hookrightarrow \mathbb{P}^1_+ \hookrightarrow P$, respectively.

**Lemma 9.6.** *Let $\mathcal{L}$ be the universal family on $\mathbb{P}^1 \times P$. Then*
$$\det \mathcal{L}|_{\mathfrak{D}\times P} \simeq O_{\mathfrak{D}\times P}(\Delta_- + \Delta_+) \otimes p_2^* \delta'.$$

*Proof.* We have a canonical map $p_1^* O_{\mathbb{P}^1}(-2) \to \mathcal{L}$ (recall the modular description of $P$). On the other hand, we have an adjunction morphism $p_2^*\delta' \to \mathcal{L}$ (recall that the fiber of $p_2^*\delta'$ is $H^0(\mathbb{P}^1, L)$). These maps give rise to a map $p_1^* O_{\mathbb{P}^1}(-2) \otimes p_2^*\delta' \to \det \mathcal{L}$, and it vanishes exactly over the graph of $\wp$. Restricting to $\mathfrak{D} \times P$ we obtain the result. □

Note that $\mathbb{C}[\mathfrak{D}]^\times$-torsors on a scheme $Y$ are the same as $\mathfrak{D}$-families of line bundles on $Y$ (i.e., line bundles on $\mathfrak{D} \times Y$). Indeed, a line bundle on $\mathfrak{D} \times Y$ is the same as a rank one locally free module over $\mathbb{C}[\mathfrak{D}] \otimes_{\mathbb{C}} O_Y$. Such modules are in one-to-one correspondence with torsors over the sheaf $(\mathbb{C}[\mathfrak{D}] \otimes_{\mathbb{C}} O_Y)^\times$.

Thus $\eta_{univ}$ and $\eta'_{univ}$ can be viewed as line bundles on $\mathfrak{D}\times\mathcal{B}un(-1)$. Clearly, $\eta_{univ}$ is a subbundle of $\mathcal{L}|_{\mathfrak{D}\times\mathcal{B}un(-1)}$, and $\eta'_{univ} = (\mathcal{L}|_{\mathfrak{D}\times\mathcal{B}un(-1)})/\eta_{univ}$.

**Proposition 9.7.** (*a*) *The pullback of $\eta_{univ}$ to $\mathfrak{D} \times P$ is $O_{\mathfrak{D}\times P}(\Delta_+)$.*
(*b*) *The pullback of $\eta'_{univ}$ to $\mathfrak{D} \times P$ is $p_2^*(O_P(2(\infty) - \sum n_i x_i^-)) \otimes O_{\mathfrak{D}\times P}(\Delta_-)$.*

*Remark* 9.8. The asymmetry is due to the choice of one of two torsors $P \to \mathcal{B}un(-1)$.

*Proof.* Note that $\pi^*\eta_{univ} \otimes \pi^*\eta'_{univ} = \det \mathcal{L}|_{\mathfrak{D}\times P}$, thus (a) follows from (b) and Lemma 9.6. Let us prove (b). The proof is essentially a family version of Lemma 9.4.

As in the proof of Lemma 9.6 we get a map $\bar{\delta} := p_2^*\delta' \to \mathcal{L}$. Restricting this to $\mathfrak{D} \times P$ and composing with the natural projection we get a map
$$\bar{\delta}|_{\mathfrak{D}\times P} \to \pi^*\eta'_{univ}.$$
We need to show that it vanishes exactly on $\Delta_-$. Clearly, this map vanishes on $S \subset \mathfrak{D} \times P$ if and only if $\bar{\delta} \to \mathcal{L}$ factors through $\eta_{univ}$ over $S$. One checks that this happens if and only if $\bar{\delta}(-(\mathfrak{D} \times P)) \to \mathcal{L}(-(\mathfrak{D} \times P)) \to \mathcal{L}_{\eta_{univ}}$ vanishes over $S$. Let us show that $S \subset \Delta_-$ in this case (we leave the converse to the reader).

We see that the rank of $\varphi' : \bar{\delta}(-(\mathfrak{D} \times P)) \oplus p_1^* O_{\mathbb{P}^1}(-2) \to \mathcal{L}_{\eta_{univ}}$ drops on $S$. Recall the modular definition of $\wp$: its graph is given by the scheme, where the rank of $\varphi'$ drops (cf. proof of Proposition 4.10, Step 5). Thus $S \subset \Delta_- \cup \Delta_+$. But the kernel of $\varphi'$ is $\bar{\delta}(-(\mathfrak{D} \times P))$, thus in fact $S \subset \Delta_-$. □

Now let us be explicit about what we need to calculate: $\pi^*\eta_{univ}$ and $\pi^*\eta'_{univ}$ correspond to classes $[\pi^*\eta_{univ}], [\pi^*\eta'_{univ}] \in H^1(\mathfrak{D}\times P, O^\times_{\mathfrak{D}\times P})$. There is a natural map $\mathbf{dlog} : O^\times_{\mathfrak{D}\times P} \to p_2^*\Omega_P : f \mapsto f^{-1}\mathbf{d}_P f$. Applying this map to $[\pi^*\eta_{univ}]$ and $[\pi^*\eta'_{univ}]$ we get elements of $H^1(P,\Omega_P)\otimes_{\mathbb{C}} O_{\mathfrak{D}}$. The TDO ring $\pi^\bullet\mathcal{D}_{\mathcal{B}un(-1),\alpha}$ corresponds to an element of $H^1(P,\Omega_P)$ given by

$$\langle \mathbf{dlog}[\pi^*\eta_{univ}], (\alpha_i^+)\rangle + \langle \mathbf{dlog}[\pi^*\eta'_{univ}], (\alpha_i^-)\rangle,$$

where

$$\langle\cdot,\cdot\rangle : H^1(P,\Omega_P)\otimes O_{\mathfrak{D}}\otimes O^\vee_{\mathfrak{D}} \to H^1(P,\Omega_P).$$

Choose local parameters $z_i$ at $x_i \in \mathbb{P}^1$. Then we obtain an isomorphism $O_{\mathfrak{D}} = \prod_i \mathbb{C}[w_i]/w_i^{n_i}$. Recall the description of $H^1(P,\Omega_P)$ given in Lemma 2.1. An easy calculation shows that

$$\begin{aligned}
\mathbf{dlog}\left(p_2^*(2(\infty) - \sum n_i x_i^-)\right) &= (0, n_i\,\mathbf{d}z_i/z_i)\otimes 1_{\mathfrak{D}},\\
\mathbf{dlog}(O_{\mathfrak{D}\times P}(\Delta_-)) &= \left(1_{\mathfrak{D}}, -\frac{\mathbf{d}z_i}{z_i - w_i}\right),\\
\mathbf{dlog}(O_{\mathfrak{D}\times P}(\Delta_+)) &= \left(1_{\mathfrak{D}}, \frac{\mathbf{d}z_i}{z_i - w_i}\right)
\end{aligned}$$

($\frac{\mathbf{d}z_i}{z_i-w_i}$ should be expanded in the powers of $w_i$). Further,

$$\begin{aligned}
\langle (0, n_i\,\mathbf{d}z_i/z_i)\otimes 1_{\mathfrak{D}}, (\alpha_i^-)\rangle &= (0, n_i\lambda\,\mathbf{d}z_i/z_i),\\
\left\langle \left(1_{\mathfrak{D}}, -\frac{\mathbf{d}z_i}{z_i - w_i}\right), (\alpha_i^-)\right\rangle &= (\lambda, -\alpha_i^-),\\
\left\langle \left(1_{\mathfrak{D}}, \frac{\mathbf{d}z_i}{z_i - w_i}\right), (\alpha_i^+)\right\rangle &= \left(\sum_i \operatorname{res}\alpha_i^+, \alpha_i^+\right).
\end{aligned}$$

Note that collections $(\alpha_i^\pm)$ in the left-hand side are viewed as elements of $O^\vee_{\mathfrak{D}}$, while in the right-hand side they are polar parts of 1-forms.

Applying the previous proposition and recalling that $\lambda + \sum_i \operatorname{res}\alpha_i^+ = -d$, we see that the element of $H^1(P,\Omega_P)$ corresponding to $\pi^\bullet\mathcal{D}_{\mathcal{B}un(-1),\alpha}$ is

$$(-d, \alpha_i^+ - \alpha_i^- + n_i\lambda\,\mathbf{d}z_i/z_i).$$

It remains to notice that $\beta_i^\pm$ correspond to the same element of $H^1(P,\Omega_P)$, cf. Lemma 2.1.

9.4. **Step 3: $\mathcal{M}_\alpha \simeq \mathcal{M}_\beta$.** This isomorphism is provided by Katz's middle convolution. It is defined in [Kat] in the settings of $l$-adic sheaves; see [Sim4] or [Ari1] for the settings of de Rham local systems. Here is an explicit description of the isomorphism.

Fix $\infty \in \mathbb{P}^1 - \mathfrak{D}$. There is a unique 1-form $\alpha$ on $\mathbb{P}^1 - \mathfrak{D} - \{\infty\}$ such that $\alpha + \alpha_i^-$ is non-singular at $x_i$ and $\alpha$ has a pole of order one at $\infty$. Similarly, there is a unique 1-form $\beta$ on $\mathbb{P}^1 - \mathfrak{D} - \{\infty\}$ such that $\beta + \beta_i^-$ is non-singular at $x_i$ and $\beta$ has a pole of order one at $\infty$. Note that $\operatorname{res}_\infty \alpha = \sum_i \operatorname{res}\alpha_i^- = \lambda$ and $\operatorname{res}_\infty\beta = -\lambda$.

Fix $(L,\nabla) \in \mathcal{M}_\alpha$. The connection

$$\nabla + \alpha : L \to L\otimes\Omega_{\mathbb{P}^1}(\mathfrak{D} + (\infty))$$

has formal type $(0, \alpha_i^+ - \alpha_i^-)$ at $x_i$. Let $\tilde{L} \subset L$ be the largest subsheaf such that

$$(\nabla + \alpha)(\tilde{L}) \subset L\otimes\Omega_{\mathbb{P}^1}(\infty).$$

Explicitly, $\tilde{L}$ is the modification of $L$ with respect to one of two parabolic structures on $L$ induced by $\nabla$. Precisely, this parabolic structure $\eta$ is such that the polar part of $\nabla$ induces multiplication by $\alpha_-$ on $\eta$ (cf. Corollary 4.4).

Consider on $\mathbb{P}^1 \times \mathbb{P}^1$ the differential 1-form $\lambda \mathbf{d}\log(x-y)$, where $x$ and $y$ are the coordinates on the first and second factors, respectively. The preimage $p_1^* L$ carries a flat meromorphic connection $p_1^*\nabla$; let us equip $p_1^* L$ with the flat meromorphic connection

$$p_1^*\nabla + p_1^*\alpha + \lambda \mathbf{d}\log(x-y).$$

Denote the 'horizontal' and 'vertical' parts of this connection by $\nabla_x$ and $\nabla_y$. We then obtain an anti-commutative square

$$\begin{array}{ccc} p_1^*\tilde{L} & \xrightarrow{\nabla_x} & p_1^* L \otimes p_1^*\Omega_{\mathbb{P}^1}(\Delta) \\ {\scriptstyle\nabla_y}\big\downarrow & & {\scriptstyle\nabla_y}\big\downarrow \\ p_1^*\tilde{L} \otimes p_2^*\Omega_{\mathbb{P}^1}(\Delta + \mathbb{P}^1 \times \{\infty\}) & \xrightarrow{\nabla_x} & p_1^* L \otimes p_1^*\Omega_{\mathbb{P}^1} \otimes p_2^*\Omega_{\mathbb{P}^1}(2\Delta + \mathbb{P}^1 \times \{\infty\}). \end{array} \tag{9.3}$$

Consider the complex

$$\mathcal{F}^\bullet := (p_1^*\tilde{L} \xrightarrow{\nabla_x} p_1^* L \otimes p_1^*\Omega_{\mathbb{P}^1}(\Delta))$$

of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. The differential $\nabla_x$ is $p_2^{-1}O_{\mathbb{P}^1}$-linear, so the direct image $Rp_{2,*}\mathcal{F}^\bullet$ makes sense as an object in the derived category of $O_{\mathbb{P}^1}$-modules. It is easy to see that $R^0 p_{2,*}\mathcal{F}^\bullet = R^2 p_{2,*}\mathcal{F}^\bullet = 0$. Now the Euler characteristic argument shows that $Rp_{2,*}\mathcal{F}^\bullet[1]$ is a locally free $O_{\mathbb{P}^1}$-module of rank two; let us denote it by $E$.

Similarly, consider the complex

$$\mathcal{F}^\bullet(\Delta) = (p_1^*\tilde{L}(\Delta) \xrightarrow{\nabla_x} p_1^* L \otimes p_1^*\Omega_{\mathbb{P}^1}(2\Delta)).$$

Then $Rp_{2,*}\mathcal{F}^\bullet(\Delta)[1]$ is a locally free $O_{\mathbb{P}^1}$-module of rank two; let us denote it by $\tilde{E}$.

The natural morphism $\mathcal{F}^\bullet \hookrightarrow \mathcal{F}^\bullet(\Delta)$ induces a homomorphism $\iota : E \to \tilde{E}$. Recall that

$$O_{\mathbb{P}^1\times\mathbb{P}^1}(k\Delta)/O_{\mathbb{P}^1\times\mathbb{P}^1}((k-1)\Delta) \approx (\iota_\Delta)_* \mathcal{T}_{\mathbb{P}^1}^{\otimes k}.$$

Thus we have an exact sequence of complexes

$$0 \to \mathcal{F}^\bullet \to \mathcal{F}^\bullet(\Delta) \to (\iota_\Delta)_*(\tilde{L} \otimes \mathcal{T}_{\mathbb{P}^1} \to L \otimes \mathcal{T}_{\mathbb{P}^1}) \to 0.$$

One checks that the differential in the rightmost complex is induced by the natural inclusion $\tilde{L} \hookrightarrow L$. Thus $\iota$ is an embedding, and $\mathrm{Coker}(\iota) \simeq O_{\mathfrak{D}}$. We can thus identify $\tilde{E}$ with an upper modification of $E$.

Finally, note that diagram (9.3) provides a $\mathbb{C}$-linear map $\nabla_E : E \to \tilde{E} \otimes \Omega_{\mathbb{P}^1}(\infty)$. Clearly, $\nabla_E$ satisfies the Leibnitz identity. We view $\nabla_E$ as a connection on $E$ with poles at $\mathfrak{D} \cup \{\infty\}$.

**Proposition 9.9.** *The formal type of $\nabla_E$ at $x_i$ is $(0, \beta_i^+ - \beta_i^-)$, and the residue of $\nabla_E$ at $\infty$ is $-\lambda$. In other words, $(E, \nabla_E - \beta) \in \mathcal{M}_\beta$. The correspondence*

$$(L, \nabla) \mapsto (E, \nabla_E - \beta)$$

*is an isomorphism $\mathcal{M}_\alpha \xrightarrow{\sim} \mathcal{M}_\beta$.*

We shall prove a slightly weaker statement, which is sufficient for our purposes. Namely, we prove that $(E, \nabla_E)$ has the described formal types after a modification.

*Proof.* Let $\mathcal{D}_{\mathbb{P}^1,\lambda}$ be the TDO ring corresponding to $\lambda \in \mathbb{C} = H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1})$. For $(L,\nabla) \in \mathcal{M}_\alpha$ consider the $\mathcal{D}_{\mathbb{P}^1,\lambda}$-module $\jmath_{!*}(L,\nabla+\alpha)$, where $\jmath : \mathbb{P}^1 - (\mathfrak{D}\cup\infty) \hookrightarrow \mathbb{P}^1$ is the natural inclusion. Note that it has no singularity at $\infty$ because of the twist by $\lambda$.

As explained in [Ari1, §6.3], the Katz–Radon transform gives an equivalence $\mathfrak{R} : \mathcal{D}_{\mathbb{P}^1,\lambda} - \mathrm{mod} \to \mathcal{D}_{\mathbb{P}^1,-\lambda} - \mathrm{mod}$ and it is easy to see that it is compatible with our construction in the sense that

$$\mathfrak{R}(\jmath_{!*}\jmath^*(L,\nabla+\alpha)) = \jmath_{!*}\jmath^*(E,\nabla_E).$$

(The proof is similar to [Ari2, Lemma 13].)

Let $\Phi_{x_i}$ be the functor of vanishing cycles as defined in [Ari1]. We get

$$\begin{aligned}\Phi_{x_i}(\jmath_{!*}\jmath^*(E,\nabla_E)) = \Phi_{x_i}\mathfrak{R}(\jmath_{!*}\jmath^*(L,\nabla+\alpha)) = \mathfrak{R}(x_i,x_i)\Phi_{x_i}\jmath_{!*}\jmath^*(L,\nabla+\alpha) = \\ \mathfrak{R}(x_i,x_i)(O_{\dot{D}}, \mathbf{d}+\alpha_i^+ - \alpha_i^-) = (O_{\dot{D}}, \mathbf{d}+\alpha_i^+ - \alpha_i^- + n_i\lambda).\end{aligned}$$

Here $\mathfrak{R}(x_i,x_i)$ is the local Katz–Radon transform, the second equality is [Ari1, Corollary 6.11], the last equality is [Ari1, Theorem C]. Now it is easy to see that $\jmath_{!*}\jmath^*(E,\nabla_E)$ has required singularities.

It follows that the formal type of $\nabla_E$ at $x_i$ is $(m_i, \beta_i^+ - \beta_i^- + m_i')$, where $m_i$, and $m_i'$ are integers. Thus $(E,\nabla_E-\beta)$ becomes a connection in $\mathcal{M}_\beta$ after a suitable modification.

Note that the construction of $(E,\nabla_E-\beta)$ works in families as well. Since formal normal forms of connections exist in families, after a suitable modification we get a morphism $\mathcal{M}_\alpha \to \mathcal{M}_\beta$. If we do the same construction with $\alpha$ and $\beta$ switched and use the inverse Katz–Radon transform, we get a morphism $\mathcal{M}_\beta \to \mathcal{M}_\alpha$. It is easy to see that these morphisms are inverse to each other. □

This completes the proof of Theorem 3.

## References


[AB] Dmitry Arinkin and Roman Bezrukavnikov. Perverse coherent sheaves. arXiv:0902.0349.

[AL] Dmitry Arinkin and Sergey Lysenko. Isomorphisms between moduli spaces of $SL(2)$-bundles with connections on $\mathbb{P}^1 \setminus \{x_1,\dots,x_4\}$. *Math. Res. Lett.*, 4(2–3):181–190, 1997.

[Alp] Jarod Alper. Good moduli spaces for Artin stacks. arXiv:0804.2242.

[Ari1] Dmitry Arinkin. Fourier transform and middle convolution for irregular D-modules. arXiv:0808.0699.

[Ari2] Dmitry Arinkin. Orthogonality of natural sheaves on moduli stacks of SL(2)-bundles with connections on $\mathbb{P}^1$ minus 4 points. *Selecta Math. (N.S.)*, 7(2):213–239, 2001.

[Ati] Michael F. Atiyah. Complex analytic connections in fibre bundles. *Trans. Amer. Math. Soc.*, 85:181–207, 1957.

[BB1] Alexander Beilinson and Joseph Bernstein. A proof of Jantzen conjectures. In *I. M. Gelfand Seminar*, volume 16 of *Adv. Soviet Math.*, pages 1–50. Amer. Math. Soc., Providence, RI, 1993.

[BB2] Roman Bezrukavnikov and Alexander Braverman. Geometric Langlands correspondence for D-modules in prime characteristic: the GL($n$) case. *Pure Appl. Math. Q.*, 3(1):153–179, 2007. Special Issue: In honor of Robert D. MacPherson. Part 3.

[BK] Igor Burban and Bernd Kreussler. Fourier-Mukai transforms and semi-stable sheaves on nodal Weierstraß cubics. *J. Reine Angew. Math.*, 584:45–82, 2005.

[BZN] David Ben-Zvi and Thomas Nevins. From solitons to many-body systems. *Pure Appl. Math. Q.*, 4(2, part 1):319–361, 2008.

[CB] William Crawley-Boevey. Indecomposable parabolic bundles and the existence of matrices in prescribed conjugacy class closures with product equal to the identity. *Publ. Math. Inst. Hautes Études Sci.*, (100):171–207, 2004.

[Fed] Roman M. Fedorov. Algebraic and hamiltonian approaches to isostokes deformations. *Transform. Groups*, 11(2):137–160, 2006.

[FG] Edward Frenkel and Benedict Gross. A rigid irregular connection on the projective line. *Annals of Mathematics*. arXiv:0901.2163.

[Fre1] Edward Frenkel. Recent advances in the Langlands program. *Bull. Amer. Math. Soc. (N.S.)*, 41(2):151–184, 2004.

[Fre2] Edward Frenkel. Ramifications of the geometric Langlands program. In *Representation theory and complex analysis*, volume 1931 of *Lecture Notes in Math*, pages 51–135. Springer, Berlin, 2008.

[EGAII] Alexander Grothendieck. Éléments de géométrie algébrique. II. Étude globale élementaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, (8):222pp, 1961.

[EGAIII] Alexander Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, (11):167, 1961.

[EGAIV] Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II. *Inst. Hautes Études Sci. Publ. Math.*, (24):231pp, 1965.

[Har] Robin Hartshorne. *Algebraic geometry*. Number 52 in Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1977.

[Kat] Nicholas M. Katz. *Rigid Local Systems*, volume 139 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.

[Lys] Sergey Lysenko. Orthogonality relations for two kinds of local systems on $\mathbb{P}^1$. Unpublished.

[GIT] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory. Third edition*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, 1994.

[Mih1] Alexandru Mihai. Sur le résidu et la monodromie d'une connexion méromorphe. *C. R. Acad. Sci. Paris Sér. A-B*, 281(12):Aii, A435–A438, 1975.

[Mih2] Alexandru Mihai. Sur les connexions méromorphes. *Rev. Roumaine Math. Pures Appl.*, 23(2):215–232, 1978.

[Mum] David Mumford. *Abelian Varieties*. Number 5 in Tata Institute of Fundamental Research Studies in Mathematics. Oxford University Press, London, 1970. Published for the Tata Institute of Fundamental Research, Bombay.

[Nev] Thomas Nevins. Mirabolic Langlands duality and the quantum Calogero-Moser system. *Transform. Groups*, 14(4):931–983, 2009.

[OO] Yousuke Ohyama and Shoji Okumura. A coalescent diagram of the Painlevé equations from the viewpoint of isomonodromic deformations. *J. Phys. A*, 39(39):12129–12151, 2006.

[Sim1] Carlos T. Simpson. Nonabelian Hodge theory. In *Proceedings of the International Congress of Mathematicians (Kyoto, 1990)*, volume I, II, pages 747–756. Math. Soc. Japan, Tokyo, 1991.

[Sim2] Carlos T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. I, II. *Inst. Hautes Études Sci. Publ. Math.*, (79, 80):47–129, 5–79, 1994, 1995.

[Sim3] Carlos T. Simpson. The Hodge filtration on nonabelian cohomology. In *Algebraic geometry—Santa Cruz, 1995*, volume 62, Part 2 of *Proc. Sympos. Pure Math.*, pages 217–281. Amer. Math. Soc., Providence, RI, 1997.

[Sim4] Carlos T. Simpson. Katz's middle convolution algorithm. *Pure Appl. Math. Q.*, 5(2):781–852, 2009. Special Issue: In honor of Friedrich Hirzebruch. Part 1.

[ST] Paul Seidel and Richard Thomas. Braid group actions on derived categories of coherent sheaves. *Duke Math. J.*, 108(1):37–108, 2001.

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